Map Graphs*

Zhi-Zhong Chen† Michelangelo Grigni‡ Christos H. Papadimitriou§

January 15, 2002

Abstract

We consider a modified notion of planarity, in which two nations of a map are considered adjacent when they share any *point* of their boundaries (not necessarily an *edge*, as planarity requires). Such adjacencies define a *map graph*. We give an NP characterization for such graphs, derive some consequences regarding sparsity and coloring, and survey some algorithmic results.

1 Introduction

1.1 Motivation and Definition

Suppose you are told that there are four planar regions, and for each pair you are told their topological relation: A is inside B, B overlaps C, C touches D on the outside, D overlaps B, D is disjoint from A, and C overlaps A. All four planar regions are "bubbles" with no holes; more precisely, they are closed disc homeomorphs. Are such regions possible? If so, we would like a model, a picture of four regions so related; if not, a proof of impossibility.

This extension of propositional logic is known as the *topological inference problem* [11]. No decision algorithm or finite axiomatization is known, although the problem becomes both finitely axiomatizable and polynomial-time decidable in any number of dimensions other than two [1, 15], for some reasonable vocabularies of topological relations. In fact, the following special case has been open since the 1960's [10]: for every pair of regions, we are only told whether the regions intersect or not. This is known as the *string graph problem*, because the input is a simple graph, and we may assume that the regions are in fact planar curves (or slightly fattened simple curves, if we insist on disc homeomorphs). In other words, we are seeking a recognition algorithm for the intersection graphs of planar curves. Until recently it was open whether this problem is even decidable; in fact it is NP-complete [13, 16, 17].

The difficulty of the string graph problem stems to a large extent from the complex overlaps allowed between regions. But many practical applications are so structured that no two regions

*A preliminary version appeared with the title "Planar Map Graphs" [6].

†Department of Mathematical Sciences, Tokyo Denki University, Hatoyama, Saitama 350-0394, JAPAN. Part of work done while visiting at University of California at Berkeley. E-mail: chen@r.dendai.ac.jp.

‡Department of Mathematics and Computer Science, Emory University, Atlanta, GA 30322. Supported by NSF Grant number CCR-9820931. E-mail: mic@mathcs.emory.edu.

§Computer Science Division, University of California at Berkeley, Berkeley, CA 94720. Supported by NSF Grant number CCR-9626361. E-mail: christos@cs.berkeley.edu.

in them overlap arbitrarily. For example, consider maps of political regions: such regions either contain one another or else they have disjoint interiors; general overlaps are not allowed.

In this paper, we consider the following special case: for every pair of regions, they are either disjoint, or they intersect only on their boundaries. Since the nations of political maps intersect in this way, we call such regions *nations*. In this paper we study *map graphs*, the intersection graphs of nations.

Definition 1.1 *Suppose G is a simple graph. A* map *of G is a function $\mathcal{M}$ taking each vertex v of G to a closed disc homeomorph $\mathcal{M}(v)$ in the sphere, such that:*

1. *For every pair of distinct vertices u and v, the interiors of $\mathcal{M}(u)$ and $\mathcal{M}(v)$ are disjoint.*
2. *Two vertices u and v are adjacent in G if and only if the boundaries of $\mathcal{M}(u)$ and $\mathcal{M}(v)$ intersect.*

If G has a map, then it is a map graph*. The regions $\mathcal{M}(v)$ are the* nations *of $\mathcal{M}$. The uncovered points of the sphere fall into open connected regions; the closure of each such region is a* hole *of $\mathcal{M}$.*

Can we recognize map graphs? This problem is closely related to planarity, one of the most basic and influential concepts in graph theory. Usually, planarity is defined as above, but with adjacency only for those pairs of nations sharing a curve segment. Planarity may also be defined in terms of maps: specifically, a planar graph has a map *such that no four nations meet at a point*.

We consider a natural restriction on maps and map graphs. Suppose we restrict our map so that *no more than k nations meet at a point*; we call this a *k-map*, and the corresponding graph is a *k-map graph*. In particular the ordinary planar graphs are the 2-map graphs; in fact all 3-map graphs are also 2-map graphs, as argued below.

In our figures we draw a map by projecting one point of the sphere to infinity; we always choose a point that is not on a nation's boundary.

1.2 Summary of Results

In Section 2 we present a characterization of map graphs G as the "half-squares" of planar bipartite graphs H (Theorem 2.2); we call H a *witness* of G. This characterization implies that map graph recognition is in NP (Corollary 2.4), and also some sparsity results for k-map graphs. In Section 3 we characterize all possible clique maps (Theorem 3.1), and we show that a map graph has a linear number of maximal cliques (Theorem 3.2). In Section 4 we consider the coloring of k-map graphs. We briefly survey the state of recognition algorithms in Section 5.

1.3 Examples

We give three examples. First, consider the adjacency graph of the United States in Figure 1.1; this is a 4-map graph. It is not planar, since the "four corners" states (circled) form a K_4, which is part of a K_5-minor.

Second, consider the 17-nation 4-map in Figure 1.2(1). Let G be its map graph. At the end of Section 2, we show that after deleting the edge $\{6,7\}$ from G, the result is not a map graph. This example demonstrates that the 4-map graph property is not monotone.

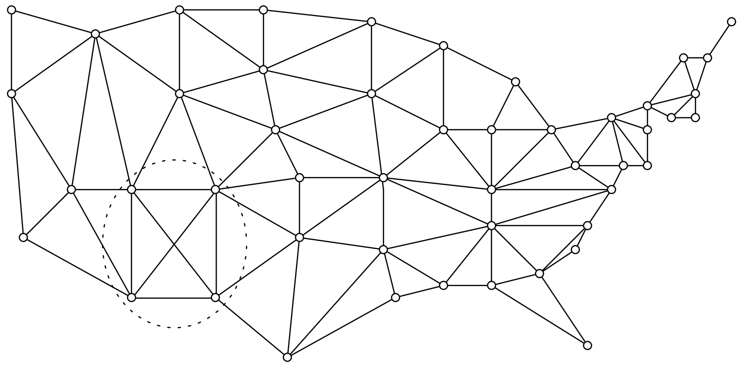
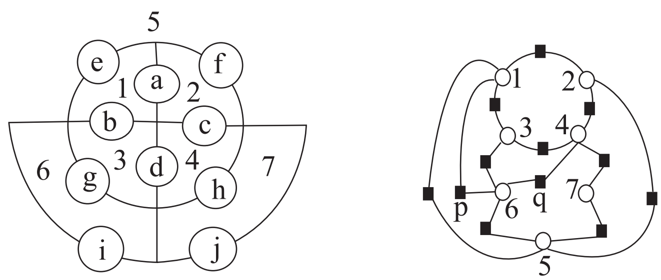


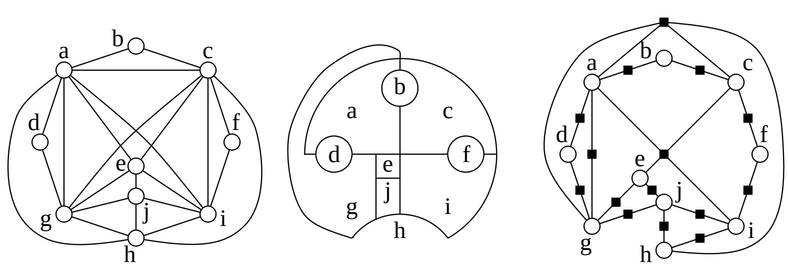
Figure 1.1: The USA map graph.



(1) (2)

Figure 1.2: An example 4-map, and a subgraph of its witness.

Third, consider Figure 1.3: part (1) is a map graph, part (2) is a 4-map of the graph, and part (3) is a corresponding witness (as defined in Section 2). The graph has a mirror symmetry exchanging a with c, but the map and the witness do not. In fact a careful analysis shows that no map or witness of this graph has such a symmetry, and so a layout algorithm must somehow "break symmetry" to find a map for this graph.



(1) (2) (3)

Figure 1.3: A symmetric map graph, a map, and a witness.

2 A Characterization

We characterize map graphs in terms of planar bipartite graphs. For a graph H and a subset A of its vertices, let $H[A]$ denote the subgraph of H induced by A. Let H^2 denote the square of H, that is the simple graph with the same vertex set, where vertices are adjacent whenever they are connected by a two-edge path in H. We represent a bipartite graph as $H = (A, B; E)$, where the vertices are partitioned into the independent sets A and B, and E is the set of edges.

Definition 2.1 *Suppose $H = (A, B; E)$ is a bipartite graph. Then $H^2[A]$ is the* half-square *of H. That is, the half-square has vertex set A, where two vertices are adjacent exactly when they have a common neighbor in B.*

Theorem 2.2 *A graph G is a map graph if and only if it is the half-square of some planar bipartite graph H.*

Proof: For the "only if" part, suppose G is a map graph. Let $\mathcal{N}$ be the set of nations in a map of G; for convenience we identify the n vertices of G with the corresponding nations in $\mathcal{N}$.

Consider a single nation R. Clearly at most $n - 1$ boundary points will account for all the adjacencies of R with other nations, and so a finite collection $\mathcal{P}$ of boundary points witnesses all the adjacencies among the nations in $\mathcal{N}$.

In each nation R we choose a representative interior point, and connect it with edges through the interior of R to the points of $\mathcal{P}$ bounding R. In this way we construct a planar embedding of the bipartite graph $H = (\mathcal{N}, \mathcal{P}; E')$, such that any two nations R_1 and R_2 intersect on their boundaries if and only if they have distance two in H. In other words, G is the half-square $H^2[\mathcal{N}]$.

For the "if" part, given a bipartite planar graph $H = (\mathcal{N}, \mathcal{P}; E')$, we embed it in the plane. By drawing a sufficiently thin star-shaped nation around each $R \in \mathcal{N}$ and its edges in E', we obtain a map for $H^2[\mathcal{N}]$. $\square$

When graphs G and H are related as above, H acts as a proof that G is a map graph. We call H a *witness* for G, and we call the vertices in $\mathcal{P}$ the *points* of the witness; such points are displayed as squares in the example Figure 1.3(3). The above argument shows that H has at most quadratic size, but we can do better.

Lemma 2.3 *If G is a map graph with n vertices, then it has a witness H with $O(n)$ vertices and edges.*

Proof: Construct H as above. A point $p \in \mathcal{P}$ is *redundant* if all pairs of its neighbors are also connected through other points of $\mathcal{P}$. Deleting a redundant point does not change the half-square; we repeat this until H has no redundant points.

Consider a drawing of H. For each $p \in \mathcal{P}$, we choose a pair of nations R_1 and R_2 connected only by p. Remove each p and its arcs, and replace them by a single arc from R_1 to R_2. In this way, we draw a simple planar subgraph H' of G with edge set $\mathcal{P}$ and vertex set $\mathcal{N}$. Hence $|\mathcal{P}| \leq 3n - 6$, and H has less than $4n$ vertices.

Since H is simple and bipartite, by Euler's formula it has at most $2(n + |\mathcal{P}|) - 4$ edges, which is less than $8n$. $\square$

In particular, a map graph has a witness which may be checked in linear time [12], and so we have:

Corollary 2.4 *The recognition problem for map graphs is in NP.*

Corollary 2.5 *For $k \geq 3$, k-map graphs are those with witnesses H such that every point has degree at most k. Moreover, a k-map graph with n vertices has $O(kn)$ edges.*

Proof: The first assertion is clear from the arguments in Theorem 2.2. To prove the second, first note that by Lemma 2.3, each n-vertex map graph G has a witness H with less than $8n$ edges. So, some nation R has degree at most 7 in H, and consequently R has degree at most $7(k-1)$ in the map graph. Now we delete R, and prove our edge bound by induction on n. □

Remark: The first author [5] has improved the above edge bound to $kn - 2k$.

We conclude with some further simple consequences of Theorem 2.2:

- In the witness graph, a point of degree three may be replaced by three points of degree two. Consequently, 3-map graphs are 2-map graphs (planar graphs).
- If G has no clique of size four, then it is a map graph if and only if it is a planar graph.
- A map graph may contain cliques of arbitrary size.
- From the previous two remarks, it is clear that the "map graph" property is not monotone, and hence cannot be characterized by forbidden subgraphs or minors.

Regarding the last point, we can also show a stronger example:

Claim 2.6 *There is a 4-map graph G with an edge e such that $G - e$ has no map.*

Proof: Let G be the graph realized by the 4-map in Figure 1.2(1), and let H be a witness of G. We may assume that H has no point of degree 1. Since each nation $v \in \{a, \ldots, j\}$ has exactly two neighbors in G, the degree of v in H is either 1 or 2. Suppose that the degree of some $v \in \{a, \ldots, j\}$ in H is 2. We only consider the case where $v = a$; the other cases are similar. Let p_1 and p_2 be the two neighbors of a in H. Other than a, only nations 1 and 2 can be adjacent to p_1 or p_2 in H. By this and the fact that nations 1 and 2 are adjacent to each other in G, it follows that H remains to be a witness of G even after modifying it by identifying p_1 and p_2 (note that the degree of a in H becomes 1 after this modification). So, we may assume that the degree of a in H is 1. This assumption can be made for each $v \in \{a, \ldots, j\}$.

Let H' be the bipartite graph in Figure 1.2(2), and let H'' be the graph obtained from H' by deleting points p and q and their incident edges. By the above assumption, H'' must be an induced subgraph of H (for example, the commom neighbor of nations 1 and 2 in H'' is the unique neighbor of nation a in H). In turn, by the existence of edges $\{1, 6\}$ and $\{4, 6\}$ in G and the planarity of H, H' must be a (not necessarily induced) subgraph of H. Now, by Figure 1.2(2) and planarity, point q must also be adjacent to nations 3 and 7 in H. Our argument so far did not depend on edge $e = \{6, 7\}$ in G, but now this edge has been forced by considering other edges. So in other words, $G - e$ is not a map graph. □

3 Cliques in Map Graphs

Suppose G is the clique K_n, then it may be realized in the following ways, corresponding to the four parts of Figure 3.1:

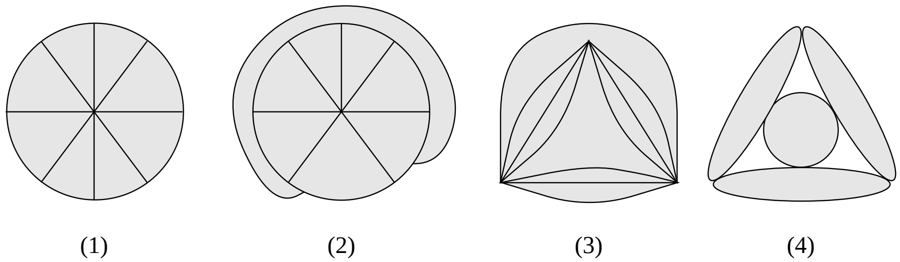


Figure 3.1: Cliques in map graphs.

(1) The n nations share a single boundary point (called the *center* of G). We call this the *pizza*.

(2) Some $n-1$ nations share a single boundary point (called the *center* of G), and the one remaining nation (called the *crust* of G) is arbitrarily connected to them at other points. We call this the *pizza-with-crust*.

(3) If $n \geq 6$, there may be three points supporting all adjacencies in the clique, with at most $n-2$ nations at any one point. In particular, there are at most two nations adjacent to all three of the points. We call this the *hamantasch*.

(4) A clique with all boundary points of degree two; that is, an ordinary planar clique. Since the planar K_2 (edge) and K_3 (triangle) are a pizza and pizza-with-crust respectively, the only new clique to list here is the planar K_4, which we call the *rice-ball*.

Theorem 3.1 *A map graph clique must be one of the above four types.*

Proof: Given a map $\mathcal{M}$ of the clique $G = K_n$, we construct a witness graph $H = (\mathcal{N}, \mathcal{P}; E)$ as in Theorem 2.2, being sure to include all intersection points of three or more nations in $\mathcal{P}$. By Euler's formula there are $O(n)$ such points, so we can still make $\mathcal{P}$ finite. Now it suffices to show that the witness H is one of the four types listed above.

Let d be the maximum degree of all points $p \in \mathcal{P}$. If $n = d$, we have a pizza. If $n = d+1$, we have a pizza-with-crust. So we may assume $n \geq d+2$. If $d \leq 3$, then the map graph is planar; since K_5 is not planar, this forces $n = 4$ and $d = 2$, the rice-ball. We now assume $d \geq 4$.

Pick point p_1 of maximum degree d, and nations x and y not adjacent to p_1. Consider the set $\mathcal{P}'$ of all points connecting x or y to the nations around p_1. We claim that there is a point $p_2 \in \mathcal{P}'$ connecting x, y, and at least two nations adjacent to p_1; otherwise, by drawing arcs through the points of $\mathcal{P}'$, we could get a planar $K_{d,2}$ with the d nations on a common face, which is impossible. Since p_1 has maximum degree, there are also two nations adjacent to p_1 but not p_2. In summary, the following three disjoint sets each contain at least two nations:

$$\begin{aligned}
\mathcal{N}_1 &= \{R \in \mathcal{N} \mid R \text{ is adjacent to } p_2 \text{ but not } p_1\} \\
\mathcal{N}_2 &= \{R \in \mathcal{N} \mid R \text{ is adjacent to } p_1 \text{ but not } p_2\} \\
\mathcal{N}_3 &= \{R \in \mathcal{N} \mid R \text{ is adjacent to both } p_1 \text{ and } p_2\}
\end{aligned}$$

We will choose six distinct nations $R_1, R_2 \in \mathcal{N}_1$, $R_3, R_4 \in \mathcal{N}_2$, and $R_5, R_6 \in \mathcal{N}_3$; no matter how we choose, the graph H will contain the induced subgraph in Figure 3.2(1), with the cycle $C = p_1 R_5 p_2 R_6$. The graph $H' = H[\{p_1, p_2\} \cup \mathcal{N}_3]$ is a complete bipartite graph, with a planar embedding inherited from H. Each face of H' is a 4-cycle; furthermore all nations in $\mathcal{N}_1 \cup \mathcal{N}_2$ must lie inside one face, in order to be connected by other points. So, we choose $R_5, R_6 \in \mathcal{N}_3$ on this face. Then this face is bounded by C; we have an embedding with $\mathcal{N}_1 \cup \mathcal{N}_2$ inside C, and $\mathcal{N}_3 - \{R_5, R_6\}$ outside C. By an appropriate choice of nations $R_1, R_2 \in \mathcal{N}_1$ and $R_3, R_4 \in \mathcal{N}_2$, we arrive at Figure 3.2(2), the embedding of p_1, p_2, and all their edges to adjacent nations. In this figure, the three occurrences of "..." locate any other nations in $\mathcal{N}_1 \cup \mathcal{N}_2 \cup \mathcal{N}_3$.

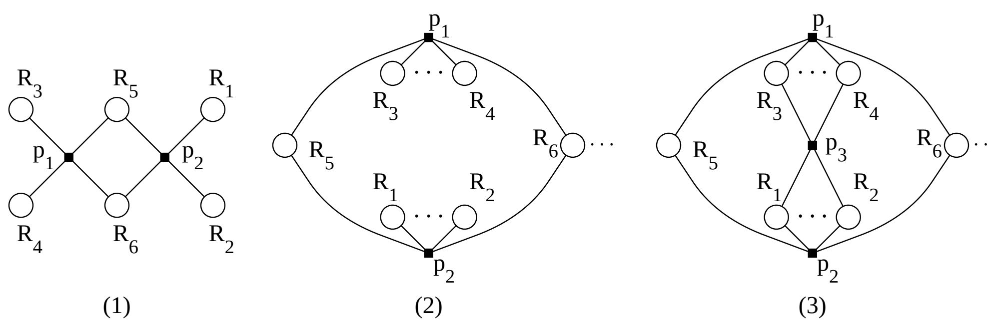


Figure 3.2: A subgraph of H, and its embedding.

There must exist a third point p_3 inside C connecting R_1 and R_4. These edges now separate $\mathcal{N}_1 - \{R_1\}$ from $\mathcal{N}_2 - \{R_4\}$, so all these nations are adjacent to p_3 as well, yielding Figure 3.2(3). This figure is not necessarily an induced subgraph, since the edges $\{R_5, p_3\}$ and $\{R_6, p_3\}$ may occur in H. But by the maximality of the degree of p_1, if exactly $i \in \{1, 2\}$ of these edges exist, then there exist i other nations $R_7 \in \mathcal{N}_3$, necessarily outside C. So, no matter whether these edges exist or not, the points p_1, p_2, and p_3 support a hamantasch on $\mathcal{N}_1 \cup \mathcal{N}_2 \cup \mathcal{N}_3$. Hence, we are done if $\mathcal{N} = \mathcal{N}_1 \cup \mathcal{N}_2 \cup \mathcal{N}_3$.

For contradiction, suppose $\mathcal{N}$ contains some nation R not adjacent to p_1 or p_2. We need to place R, and some new points and edges, in Figure 3.2(3) so that R has neighbor points connected to the other nations. However by planarity of H, if $\{R_5, p_3\}$ or $\{R_6, p_3\}$ is an edge in H, then R cannot be placed so that both R_1 and R_7 have neighbor points connected to R. Similarly, if neither $\{R_5, p_3\}$ nor $\{R_6, p_3\}$ is an edge in H, then R cannot be placed so that all of R_1, R_5, and R_6 have neighbor points connected to R.

We have shown that H is one of the four types. Since all intersection points of three or more nations in $\mathcal{M}$ are represented in H, the map $\mathcal{M}$ has the same type. We remark that the four types are not all mutually exclusive; for example $\mathcal{M}$ could contain one point of degree n and another point of degree $n - 1$. $\square$

By a careful analysis of each kind of clique, we can now show:

Theorem 3.2 *A map graph G with n vertices has at most $27n$ maximal cliques.*

Proof: We may assume that G is connected. As in Theorem 2.3, we choose a planar witness $H = (\mathcal{N}, \mathcal{P}; E')$ for G where $\mathcal{N}$ is the set of nations, $\mathcal{P}$ is the set of at most $3n - 6$ points, and E' is the set of edges.

Fix a plane embedding of H. If vertices u_1, u_2, v_1, v_2 appear in that cyclic order as distinct neighbors of some vertex w in H, then we say that the pairs $\{u_1, v_1\}$ and $\{u_2, v_2\}$ *cross* at w.

Each point can contribute to at most one maximal pizza, and so there are at most $3n-6$ maximal pizzas in G. Note that each MC_2 is a pizza.

Next, let $C_1, \ldots, C_\ell$ be the maximal cliques in G that are either non-pizza MC_3's or hamantaschen. For each hamantasch C_i, we may choose three points $p_i, q_i, r_i \in \mathcal{P}$ and three nations $a_i, b_i, c_i \in C_i$ such that $T_i = p_i a_i q_i b_i r_i c_i$ is an induced cycle in H and C_i consists of all nations adjacent to at least two of the points p_i, q_i, r_i in H. For each pair $\{a, b\}$ of nations such that some non-pizza MC_3 contains both a and b, let $s_{a,b}$ be a point in $\mathcal{P}$ that is adjacent to both a and b in H. For each non-pizza MC_3 C_i, let $C_i = \{a_i, b_i, c_i\}$, $p_i = s_{a_i,c_i}$, $q_i = s_{a_i,b_i}$, $r_i = s_{b_i,c_i}$, and T_i be the induced cycle $p_i a_i q_i b_i r_i c_i$ in H. No matter whether C_i is a non-pizza MC_3 or hamantasch, we define $f(C_i) = \{p_i, q_i, r_i\}$ and note that C_i consists of all nations adjacent to at least two points of $f(C_i)$. This implies $f(C_i) \neq f(C_j)$ for distinct C_i and C_j, because otherwise $C_i \cup C_j$ would be a larger clique.

Define H' as the simple graph with vertex set $\mathcal{P}$ and edge set $\{\{p, q\} \mid \{p, q\} \subset f(C_i)$ for some $i\}$. We claim that H' is planar. To see this, we embed H' in H by drawing the edge $\{p_i, q_i\}$ of H' through their neighbor a_i in H, and similarly for the other two edges $\{q_i, r_i\}$ and $\{p_i, r_i\}$. Towards a contradiction, assume that two edges of H' cross in the embedding. Then, by cycle symmetries, we may assume that for some distinct C_i and C_j, pairs $\{p_i, q_i\}$ and $\{p_j, q_j\}$ cross at nation $a_i = a_j$ (call it a) in H. Since the cycles T_i and T_j cross at a in H, they must cross again, sharing either another nation or a point.

Case 1: T_i and T_j share another nation but no point. By symmetry, it suffices to consider the case $b_i = b_j = b$. If C_i and C_j were both non-pizza MC_3's, then we would have $q_i = s_{a,b} = q_j$, contradicting the crossing. So at least one of C_i and C_j is a hamantasch, we suppose C_i. Then C_i has another nation a' also adjacent to p_i and q_i or else C_i would be a pizza-with-crust. Because of T_j, it must be $a' = c_j$. Now since q_i is adjacent to a, b, c_j, in order for C_j to be a non-pizza, C_j must also be a hamantasch. Then C_j has another nation a'' adjacent to p_j and q_j, but planarity of H makes this impossible.

Case 2: T_i and T_j share a point. Since T_i and T_j are induced, a is adjacent to neither r_i nor r_j, so the only possible shared point is $r_i = r_j$. In turn, $C = \{a, b_i, c_i, b_j, c_j\}$ is a clique of G. So, neither C_i nor C_j is an MC_3, and both are hamantaschen. Now as in the previous case, we find it is impossible to add a nation $a' \notin C$ between p_i and q_i and a nation $a'' \notin C$ between p_j and q_j. By this, both C_i and C_j are pizza-with-crusts, a contradiction.

By the above case-analysis, the claim holds, and H' is a simple planar graph with at most $3n-6$ vertices and at least ℓ distinct triangles. An easy exercise shows that any simple planar graph with h vertices has at most $3h$ triangles. So, $\ell \leq 9n - 18$.

Since H is embedded in the plane, for each rice-ball C, H has an induced 6-cycle containing exactly three nations of C and with the one remaining nation (called the *center*) of C lying inside C. It is clear that each nation can be the center of at most one rice-ball. So, there are at most n rice-balls.

It remains to bound the number of maximal pizza-with-crusts of size 4 or more. Fix a point p in H, let V_p denote the set of nations adjacent to p in H, and let $\mathcal{K}_p = \{C_1, \ldots, C_{\ell_p}\}$ be the maximal pizza-with-crusts with center p and size 4 or more. We claim that $\ell_p = |\mathcal{K}_p| \leq 2(2|V_p| - 3)$. This claim implies that G has at most $\sum_p (4|V_p| - 6)$ maximal pizza-with-crusts of size 4 or more. This sum equals $4|E'| - 6|\mathcal{P}|$; since $|E'| \leq 2(n + |\mathcal{P}|) - 4$ and $|\mathcal{P}| \leq 3n - 6$, the sum is less than $14n$.

Now we prove the claim. The embedding gives a cyclic clockwise order on the nations of V_p around p; this order defines "consecutive" nations and "intervals" of nations around p. For nations $u, v \in V_p$, let $[u, v]$ denote the circular interval of nations starting at u, proceeding clockwise around p, and ending at v. For each clique C_i in $\mathcal{K}_p$, let b_i be the crust of C. Since C_i is not a pizza, we can choose distinct nations $a_i, c_i \in C_i - \{b_i\}$ and distinct points $q_i, r_i \neq p$ satisfying the following three conditions:

- $T_i = pa_iq_ib_ir_ic_i$ is a simple cycle in H.
- If a nation of $C_i - \{a_i, b_i, c_i\}$ is adjacent to q_i or r_i, then it lies outside T_i, otherwise it lies inside.
- All nations of C_i lying inside the cycle T_i lie in the interval $[a_i, c_i]$.

Denote the unordered pair $\{a_i, c_i\}$ by $g(C_i)$, and $\{q_i, r_i\}$ by $h(C_i)$. By considering such 6-cycles T_i and the planarity of H, we see that there are no cliques $C_i, C_j \in \mathcal{K}_p$ such that $g(C_i)$ and $g(C_j)$ cross at p; consequently the simple graph $G_p = (V_p, \{g(C_i) \mid C_i \in \mathcal{K}_p\})$ is outerplanar, where we use the same cyclic order on V_p for the outerplanar embedding. Since G_p is simple outerplanar, it can have at most $2|V_p| - 3$ edges. Thus, to prove $|\mathcal{K}_p| \leq 2(2|V_p| - 3)$, it suffices to prove that each edge of G_p equals $g(C_i)$ for at most two $C_i \in \mathcal{K}_p$.

For contradiction, assume that there exist three distinct cliques $C_i, C_j, C_k \in \mathcal{K}_p$ with $g(C_i) = g(C_j) = g(C_k)$. Say that two of these cliques are *nested* if the crust of one is inside the 6-cycle of the other. We consider two cases.

Case I: There are two non-nested cliques. We may suppose that they are C_i and C_j. By planarity, the interiors of T_i and T_j are disjoint, with $a_i = c_j$ and $c_i = a_j$. Moreover, no matter whether $h(C_i) \cap h(C_j) = \emptyset$ or not, no nation of $V_p - g(C_i)$ is adjacent to both p and at least one point of $h(C_i) \cup h(C_j)$ in H. Thus, the set of nations lying inside T_i and the set of nations lying inside T_j together form a partition of $V_p - g(C_i)$. By this and the maximality of cliques in $\mathcal{K}_p$, the crust of C_k must lie inside T_i or T_j; by symmetry we suppose it lies inside T_i. On the other hand, since C_i is a maximal clique of size 4 or more, there exists a nation $x_i \in C_i - g(C_i)$ lying inside T_i. Since x_i cannot be adjacent to q_i or r_i, there is another point s_i lying inside T_i that connects x_i with b_i. So, we have a path $P_i = px_is_ib_i$ sharing only its endpoints with T_i, and bisecting the interior of T_i. Now the crust b_k must lie inside T_i, to one side or the other of P_i. To achieve $g(C_k) = g(C_i)$, b_k must be adjacent to s_i in H. But then we would see that $s_i \in h(C_k)$, and so either a_k or c_k was chosen incorrectly.

Case II: All three cliques nest. Again by planarity, the cycles T_i, T_j, T_k cannot cross. So, their interiors nest in some order; we may assume that b_k lies inside T_j, and b_j lies inside T_i. We have $a_i = a_j = a_k$ and $c_i = c_j = c_k$. Since C_j is a maximal clique, it contains some nation x_j not adjacent to b_i in G. By planarity, x_j must lie inside T_j, and there is some point s_j connecting b_j with x_j in H. We cannot have $s_j \in h(C_j)$, by the choice of $g(C_j)$; so again we have a path $P_j = px_js_jb_j$, sharing only its endpoints with T_j and bisecting its interior. Now the crust b_k must lie inside T_j, to one side or the other of P_j; the rest of the argument proceeds as in the last case. □

4 Coloring of k-map Graphs

For $k \geq 3$, Theorem 3.1 implies that the maximum clique size in a k-map graph is $\lfloor 3k/2 \rfloor$; therefore at least $\lfloor 3k/2 \rfloor$ colors are needed to color some k-map graphs. An interesting question is to ask whether $\lfloor 3k/2 \rfloor$ colors suffice to color all k-map graphs. In case $k = 3$, the answer is positive because of the famous Four Color Theorem. As Thorup observed [18], the answer is also positive for $k = 4$: 4-map graphs are all 1-planar (i.e., they can be drawn in the plane in such a way that each edge is crossed by at most one other edge), and 1-planar graphs are known to be 6-colorable [3]. However, the answer is unknown when $k \geq 5$.

Coloring of k-map graphs is closely related to cyclic coloring of planar graphs introduced by Ore and Plummer [14]. Let k be an integer larger than two. Let χ_k be the minimum number of colors sufficient to color the vertices of each planar graph G in such a way that (1) no two adjacent vertices get the same color and (2) for every face F of G whose boundary is a simple cycle consisting of at most k vertices, the vertices on the boundary of F get distinct colors. Ore and Plummer [14] proved that $\chi_k \leq 2k$. Later, Borodin [4] proved that $\chi_k \leq 2k - 3$ if $k \geq 8$. The conjecture that $\chi_k \leq \lfloor 3k/2 \rfloor$ was due to Borodin [2]. It is not difficult to see that this conjecture is true if and only if $\lfloor 3k/2 \rfloor$ colors suffice to color k-map graphs.

It also seems interesting to design efficient algorithms for coloring k-map graphs. The upper bound $kn - 2k$ [5] on the number of edges in a k-map graph implies a linear-time algorithm for coloring the vertices of a given k-map graph with $2k$ colors. Are there efficient algorithms for coloring the vertices of a given k-map graph with $2k - 1$ or fewer colors? In light of the difficulty of answering this question and the conjecture $\chi_k \leq \lfloor 3k/2 \rfloor$, it also seems interesting to ask whether it is NP-hard to decide if the vertices of a given k-map graph can be colored with $\lfloor 3k/2 \rfloor - 1$ colors. This question is open even when $k = 4$.

5 Hole-Free Map Graphs and Recognition Algorithms

Suppose that $\mathcal{M}$ is a map of a graph G and that every point of the sphere is covered by some nation of $\mathcal{M}$. Then we say that $\mathcal{M}$ is a *hole-free map* and that G is a *hole-free map graph*. If in addition $\mathcal{M}$ is a k-map, then we say that $\mathcal{M}$ is a *hole-free k-map* and that G is a *hole-free k-map graph*. Since removing one nation from a hole-free map leaves a connected set of nations, all hole-free map graphs are 2-connected; consequently, the adjacency graph of the United States in Figure 1.1 is not a hole-free map graph. On the other hand, the 4-map in Figure 1.2(1) is hole-free and hence Claim 2.6 indeed shows that there is a hole-free 4-map graph G with an edge e such that $G - e$ has no map. Moreover, as a simple consequence of Theorem 2.2, we have that hole-free map graphs are those with witnesses H such that the boundary of every face of H has exactly four or six edges.

Can we recognize map graphs efficiently? We would also like to recognize k-map graphs (for each choice of k), and hole-free k-map graphs. There is an obvious naive approach for all these recognition problems: for each maximal clique in the given graph, assert a point in the map where the nations of the clique should meet. This approach fails because there are other maps which realize a clique, as we saw in Figure 3.1.

In a preliminary version of this paper [6] we sketched a polynomial-time recognition algorithm for 4-map graphs; that result is omitted here for brevity. See [7] for a full presentation of a cubic time recognition algorithm for *hole-free* 4-map graphs. The basic idea behind this algorithm is to figure out the correct type (cf. Theorem 3.1) of each maximal clique C in the input graph G. The correct type of C is found by a case-analysis of the neighborhood structure of the nations of C in G. Before the case-analysis, certain separators of G are found and used to simplify G so that the case-analysis needs to consider only a few cases.

Thorup [18] has recently presented a polynomial-time algorithm for recognizing general map graphs. The main tools used in his algorithm are the PQ-tree data structure and dynamic programming. Thorup's result does not necessarily imply ours, since even if we are given a map realizing a map graph, it is not clear that it helps us to find a map with the additional restrictions we want (a hole-free 4-map). Also, Thorup's algorithm is complex and the exponent of its time bound polynomial is about 120, while our algorithm is more understandable and its time bound is $O(n^3)$.

6 Concluding Problems

The existing algorithms (both ours and Thorup's) are very complex. We would like faster algorithms, with simpler arguments. We would also prefer to output an efficiently verifiable witness when the input is not a map graph, so that we could have reliable "checkers" for these complex algorithms.

Naturally, we are very interested in polynomial-time algorithms for recognizing (hole-free or not) k-map graphs with $k \geq 5$. In view of the complication of our algorithm for hole-free 4-map graphs, however, new insights seem to be needed in order to make progress in this direction.

The recognition problem of map graphs is just a special topological inference problem, where each pair of regions either touch or are disjoint. One more general problem is obtained by allowing the relation between certain pairs of regions to be left unspecified (i.e., each such pair may touch or not touch). We conjecture that this generalization is NP-complete. Another generalization is obtained by allowing a region to include another region as a subregion. We conjecture that this generalization is polynomial-time solvable. Note that the inclusion relations among the regions should induce a rooted forest. The special case of this generalization where no four leaf regions meet at a point and each non-leaf region is the union of its descendant regions, can be solved by a nontrivial $O(n \log n)$-time algorithm [8]. In the real world, a non-leaf region is usually not a closed disc homeomorph; this more general problem is addressed in [9].

Finally, we point out three purely combinatorial open questions. The first question asks whether the upper bound on the number of maximal cliques given in Theorem 3.2 can be improved. The second asks whether the upper bound $kn - 2k$ on the number of edges in a k-map graph [5] can be improved. Next, consider a connected graph G with nonnegative weights on its vertices. The *weight* of a subgraph of G is the total weight of vertices in the subgraph. A *separator* of G is a subgraph H of G such that deleting the vertices of H from G leaves a graph whose connected components each have weight $\leq \frac{2W}{3}$, where W is the weight of G. The *size* of a separator H is the number of vertices in H. The first author [5] has shown that each k-map graph with n vertices has a separator of size $\leq 2\sqrt{2k(n-1)}$. The third question asks whether this bound can be improved.

References

[1] J. F. Allen. Maintaining knowledge about temporal intervals. *Communications of the ACM*, 26(11):832–843, 1983.

[2] O. V. Borodin. Solution of Ringel's problem on vertex-face coloring of plane graphs and coloring of 1-planar graphs (in Russian). *Metody Diskret. Analiz.*, 41:12-26, 1984.

[3] O. V. Borodin. A new proof of the 6 color theorem. *J. Graph Theory*, 19(4):507-522, 1985.

[4] O. V. Borodin. Cyclic coloring of plane graphs. *Discrete Math.*, 100:281-289, 1992.

[5] Z.-Z. Chen. Approximation algorithms for independent sets in map graphs. *J. Algorithms*, 41:20-40, 2001.

[6] Z.-Z. Chen, M. Grigni, and C. Papadimitriou. Planar map graphs. *Proc. 30th Ann. ACM Symp. Theory of Computing (STOC)*, 514–523, 1998.

[7] Z.-Z. Chen, M. Grigni, and C. Papadimitriou. Map graphs. Manuscript, arXiv:cs/9910013, 46 pages, 1999.

[8] Z.-Z. Chen and X. He. Hierarchical topological inference on planar disc maps. *Proc. 6th Ann. Intl. Computing and Combinatorics Conference (CoCoon)*, Vol. 1858 of *Lecture Notes in Computer Science*, pp. 115-125, 2000.

[9] Z.-Z. Chen, X. He, and M.-Y. Kao. Nonplanar topological inference and political-map graphs. *Proc. 10th Ann. ACM-SIAM Symp. Discrete Algorithms (SODA)*, 195–204, 1999.

[10] G. Ehrlich, S. Even, and R. E. Tarjan. Intersection graphs of curves in the plane. *J. Combinatorial Theory Ser. B*, 21(1):8–20, 1976.

[11] M. Grigni, D. Papadias, and C. H. Papadimitriou. Topological inference. *Proc. 14th Intl. Joint Conf. on Artificial Intelligence (IJCAI)*, 901–906, 1995.

[12] J. Hopcroft and R. E. Tarjan. Efficient planarity testing. *J. ACM*, 21(4):549–568, 1974.

[13] J. Kratochvíl. String graphs II: Recognizing string graphs is NP-hard. *J. Combinatorial Theory, Ser. B*, 52(1):67–78, 1991.

[14] O. Ore, M. D. Plummer. Cyclic coloration of plane graphs. In *Recent Progress in Combinatorics (Proc. 3rd Waterloo Conf. on Combinatorics, 1968)*, 287–293, 1969.

[15] J. Renz. A canonical model of the region connection calculus. In *Principles of Knowledge Representation and Reasoning: Proc. of the 6th Intl. Conf.*, 330–341, June 1998.

[16] M. Schaefer, D. Śtefanković. Decidability of string graphs. *Proc. 33rd Ann. ACM Symp. Theory of Computing (STOC)*, 241–246, 2001.

[17] M. Schaefer, E. Sedgwick, D. Śtefanković. Recognizing string graphs in NP. Depaul Univ. Tech. Report TR01-011, 2001.

[18] M. Thorup. Map graphs in polynomial time. *Proc. 39th Ann. IEEE Symp. Foundations of Computing (FOCS)*, 396–405, 1998.