New bounds on the edge number of a k-map graph *

Zhi-Zhong Chen †

Abstract

It is known that for every integer $k \geq 4$, each k-map graph with n vertices has at most $kn - 2k$ edges. Previously, it was open whether this bound is tight or not. We show that this bound is tight for $k = 4, 5$. We also show that this bound is not tight for large enough k (namely, $k \geq 374$); more precisely, we show that for every $0 < \epsilon < \frac{3}{328}$ and for every integer $k \geq \frac{140}{41\epsilon}$, each k-map graph with n vertices has at most $(\frac{325}{328} + \epsilon)kn - 2k$ edges. This result implies the first polynomial (indeed linear) time algorithm for coloring a given k-map graph with less than $2k$ colors for large enough k. We further show that for every positive multiple k of 6, there are infinitely many integers n such that some k-map graph with n vertices has at least $(\frac{11}{12}k + \frac{1}{3})n$ edges.

Key words. Planar graph, plane embedding, sphere embedding, map graph, k-map graph, edge number.

AMS subject classifications. 05C99, 68R05, 68R10.

Abbreviated title. Edge Number of k-Map Graphs.

1 Introduction

Chen, Grigni, and Papadimitriou [6] studied a modified notion of planar duality, in which two nations of a (political) map are considered adjacent when they share any *point* of their boundaries (not necessarily an *edge*, as planar duality requires). Such adjacencies define a *map graph*. The map graph is called a *k-map graph* for some positive integer k, if no more than k nations on the map meet at a point. As observed in [6], planar graphs are exactly 3-map graphs, and the adjacency graph of the United States is nonplanar but is a 4-map graph (see Figure 1.1).

The above definitions of map graphs and k-map graphs may be not rigorous enough. For this reason, we give a different rigorous definition. Consider a bipartite graph $B = (V, P; E_B)$ with vertex sets V, P and edge set $E_B \subseteq V \times P$. The *half-square* of B is the simple graph G with vertex

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†Department of Mathematical Sciences, Tokyo Denki University, Hatoyama, Saitama 350-0394, Japan. Email: chen@r.dendai.ac.jp.

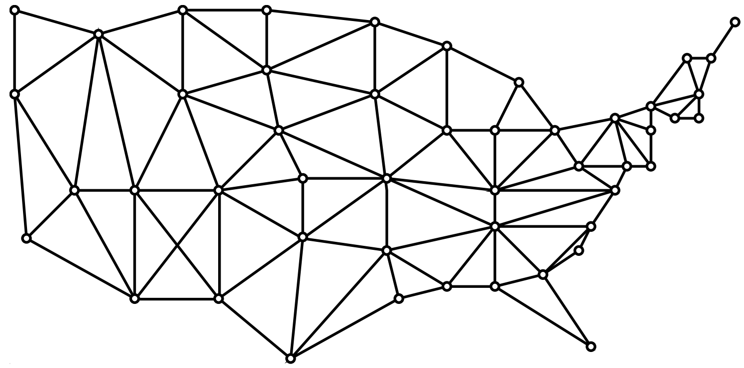


Figure 1.1: The USA map graph.

set V and edge set $E = \{\{v_1, v_2\} \mid v_1 \in V,\ v_2 \in V$, and they have a common neighbor in $B\}$. A graph is a *map graph* if it is the half square of a bipartite planar graph. For a positive integer k, a graph is a k-*map graph* if it is the half square of a bipartite planar graph $B = (V, P; E_B)$ in which each $p \in P$ is adjacent to at most k vertices in V.

1.1 History of Map Graphs

In addition to having relevance to planarity, map graphs are related to the topological inference problem which arises from theoretical studies in geographic database systems. For the details and a comprehensive survey of known results on map graphs and the topological inference problem, we refer the reader to [7] and the references therein. Recently, several interesting properties of map graphs have been found in [9, 10]. Here we only describe a brief history of research on map graphs and k-map graphs.

Coloring of k-map graphs dates back to Ore and Plummer [11], although they did not make the notion of k-map graphs explicit. In more detail, they considered the minimum number χ_k of colors sufficient to color the vertices of each planar graph G in such a way that (1) no two adjacent vertices get the same color and (2) for every face F of G whose boundary is a simple cycle consisting of at most k vertices, the vertices on the boundary of F get distinct colors. Obviously, χ_k is the minimum number of colors sufficient to color the vertices of each k-map graph. Ore and Plummer [11] proved that $\chi_k \leq 2k$. Later, Borodin [3] proved that $\chi_k \leq 2k - 3$ if $k \geq 8$. The conjecture that $\chi_k \leq \lfloor 3k/2 \rfloor$ is due to Borodin [2]. At present, the best bound on χ_k is due to Sanders and Zhao [13] who show that $\chi_k \leq \lceil 5k/3 \rceil$.

In [6] and [7], Chen *et al.* gave a simple nondeterministic polynomial-time algorithm for recognizing map graphs and investigated the structure and the number of maximal cliques in a map graph. Subsequently, Thorup [14] presented a polynomial-time algorithm for recognizing map graphs. As far as we know, Thorup's algorithm [14] for recognizing map graphs does not imply a polynomial-time recognition algorithm for k-map graphs.

As a natural extension of planar graphs, 1-planar graphs (i.e., those simple graphs that can be embedded into the plane in such a way that each edge crosses at most one other edge) have been studied extensively in the literature (see [8] and the references therein). It is obvious that a graph is 1-planar if and only if it is a subgraph of a 4-map graph. The problem of coloring 1-planar graphs using few colors has attracted very much attention [12, 11, 1, 2, 4, 8]. Ringel [12] proved that every 1-planar graph is 7-colorable and conjectured that every 1-planar graph is 6-colorable. Ringel [12] and Archdeacon [1] confirmed the conjecture for two special cases. Borodin [2] settled the conjecture in the affirmative with a lengthy proof. He [4] later came up with a relatively shorter proof. However, his proof does not lead to a linear-time algorithm for 6-coloring 1-planar graphs. Chen and Kouno [8] give a linear-time algorithm for 7-coloring 1-planar graphs.

1.2 Motivation and the New Result

The *edge number* of a graph G is the number of edges in G. This number is undoubtedly very important for many purposes (such as studying the chromatic and the independence numbers of G, determining the arboricity of G, and analyzing the time complexity of algorithms for G). For example, the most celebrated bound $3n-6$ on the edge number of an n-vertex planar graph has led to many applications in coloring and recognizing planar graphs. In particular, tight bounds on the edge number of a graph G is very useful for efficiently coloring G using as few colors as possible.

Chen [5] proved that for every integer $k \geq 4$, each k-map graph with $n \geq k$ vertices has at most $kn - 2k$ edges. It is natural to ask whether this bound is tight or not. Indeed, this question is one of the open questions asked in [7]. This bound is tight when $k = 3$, because a maximal planar graph with n vertices has $3n - 6$ vertices. In this paper, we show that this bound is tight when $k = 4$, 5. Moreover, we show that for each large enough k (namely, $k \geq 374$), this bound is not tight. More precisely, we show that for every $0 < \epsilon < \frac{3}{328}$ and for every integer $k \geq \frac{140}{41\epsilon}$, each k-map graph with $n \geq k$ vertices has at most $(\frac{325}{328} + \epsilon)kn - 2k$ edges. This result has many important consequences. For example, it implies that the arboricity of a k-map graph is at most $(\frac{325}{328} + \epsilon) \cdot k$. Moreover, it implies a linear-time algorithm for coloring a given k-map graph with at most $2 \cdot (\frac{325}{328} + \epsilon) \cdot k$ colors for every $k \geq \frac{140}{41\epsilon}$. Previously, there was no polynomial-time algorithm for coloring a given k-map graph with less than $2k$ colors, although Sanders and Zhao [13] showed that each k-map graph can be colored with at most $\lceil \frac{5}{3}k \rceil$ colors. The proof by them does not lead to a polynomial-time algorithm for coloring a given k-map graph G, because their proof requires that G be given together with its embedding into the plane (or sphere) while it is still unknown how to construct a plane (or sphere) embedding of a given k-map graph in polynomial time.

The above new upper bound (namely, $(\frac{325}{328} + \epsilon)kn - 2k$) may look too loose at first glance. Indeed, since each k-map graph can be colored with at most $\lceil \frac{5}{3}k \rceil$ colors [13], one may be tempted to show that each k-map graph with n vertices has at most $\lceil \frac{5}{6}k \rceil n$ edges. A little unexpectedly, we can show that for every positive multiple k of 6, there are infinitely many integers n such that some k-map graph with n vertices has at least $(\frac{11}{12}k + \frac{1}{3})n$ edges.

1.3 Organization of the Paper

The remainder of this paper is organized as follows. Section 2 describes several basic definitions. Section 3 details proofs of the new lower bounds. Section 4 details a proof of the new upper bound. Section 5 suggests a few interesting open problems.

2 Basic definitions

Throughout this paper, a graph may have multiple edges but no loops, while a *simple* graph has neither multiple edges nor loops. Our terminology is standard but we review the main concepts and notation. Let G be a graph. A *bridge* of G is an edge of G whose removal increases the number of connected components in G. G is *bridgeless* if it has no bridge. Let v be a vertex in G. The *degree* of v in G, denoted by $d_G(v)$, is the number of edges incident to v. Note that $d_G(v)$ may be larger than the number of neighbors of v in G because of multiple edges. Vertex v is *isolated* in G if $d_G(v) = 0$. For $U \subseteq V(G)$, the *subgraph of G induced by U* is the graph (U, E_U) where E_U consists of all $e \in E$ such that both endpoints of e are in U. A *cycle* of G is a connected subgraph C of G such that each vertex of C is incident to exactly two edges of C. A *path* of G is a simple connected subgraph P of G such that P is not a cycle and each vertex v of P satisfies $d_P(v) \leq 2$. The *length* of a cycle (respectively, path) is the number of edges on it.

A graph is *planar* if it can be embedded into the plane (respectively, sphere) so that any pair of edges can only intersect at their endpoints; a *plane* (respectively, *sphere*) graph is a planar one together with such an embedding.

Let $\mathcal{G}$ be a sphere graph. Consider the set of all points of the sphere that lie on no edge of $\mathcal{G}$. This set consists of a finite number of topologically connected regions; the closure of each such region is a *face* of $\mathcal{G}$. Let F be a face of $\mathcal{G}$. We denote by $V(F)$ (respectively, $E(F)$) the set of all vertices (respectively, edges) of G that are contained in F. The *size* of F is $|V(F)|$. $\mathcal{G}$ is *triangulated* if the boundary of each face in $\mathcal{G}$ is a cycle of length 3.

Let F_1 and F_2 be two faces of $\mathcal{G}$. F_1 and F_2 *touch* if $V(F_1) \cap V(F_2)$ is not empty. F_1 and F_2 *strongly touch* if $E(F_1) \cap E(F_2)$ is not empty. Obviously, if F_1 and F_2 strongly touch, then they touch. However, the reverse is not necessarily true. When F_1 and F_2 strongly touch in $\mathcal{G}$, *merging F_1 and F_2* is the operation of modifying $\mathcal{G}$ by deleting all edges in $E(F_1) \cap E(F_2)$.

Fix an integer $k \geq 3$. A *k-face sphere graph* is a bridgeless sphere graph with no face of size larger than k. Let $\mathcal{H}$ be a k-face sphere graph. Let F be a face of $\mathcal{H}$. F is *small* if $|V(F)| \leq \lceil \frac{k}{2} \rceil$. F is *large* if it is not small. F is *critical* if it is small and strongly touches exactly two faces of $\mathcal{H}$.

F is *dangerous* if it is critical and strongly touches a face of size less than k. We classify critical faces F into three types as follows:

- Type 1: The boundary of F is a cycle (cf. Figure 2.1(1)). (Comment: The two faces strongly touching F may or may not strongly touch.)
- Type 2: The boundary of F is formed by two vertex-disjoint cycles (cf. Figure 2.1(2)).
- Type 3: The boundary of F is formed by two edge-disjoint cycles and the two cycles share exactly one vertex (cf. Figure 2.1(3)).

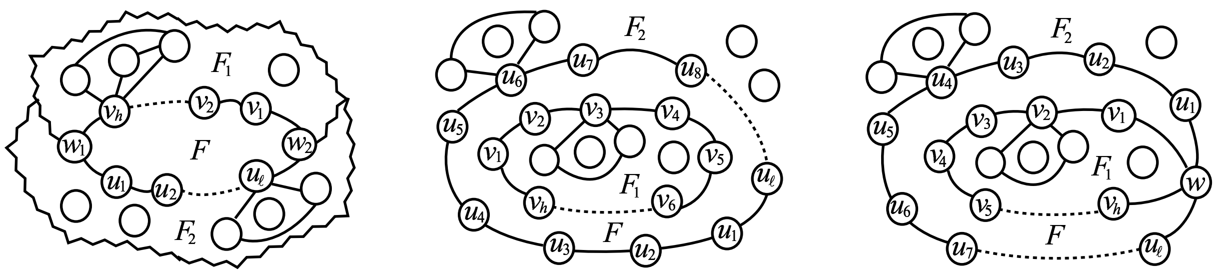


Figure 2.1: (1) A Type-1 critical face F. (2) A Type-2 critical face F. (3) A Type-3 critical face F.

A vertex v of $\mathcal{H}$ is *critical* if $d_{\mathcal{H}}(v) \geq 3$, v appears on the boundary of a critical face F of $\mathcal{H}$, and v also appears on the boundary of exactly one of the two faces strongly touching F in $\mathcal{H}$.

The *common-face graph* of $\mathcal{H}$ is the simple graph whose vertices are the vertices of $\mathcal{H}$ and whose edges are those $\{v_1, v_2\}$ such that there is a face F in $\mathcal{H}$ with $\{v_1, v_2\} \subseteq V(F)$ (see Figure 2.2 for an example).

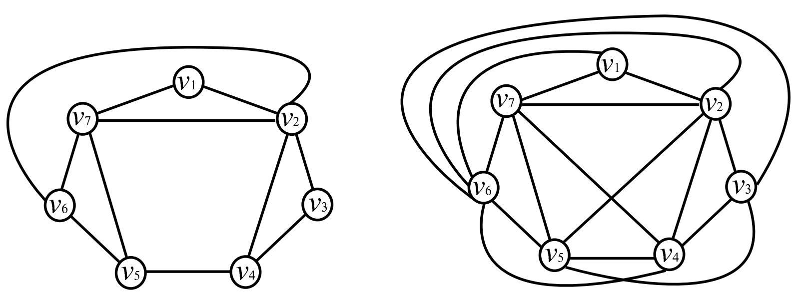


Figure 2.2: (1) A sphere graph $\mathcal{H}$. (2) The common-face graph of $\mathcal{H}$.

Fact 2.1 *The common-face graph of each triangulated sphere graph is itself and hence is planar.*

Proof. Obvious. □

Lemma 2.2 *Let k be an integer larger than 2. Then, the following statements hold:*

1. *The common-face graph of each k-face sphere graph is a k-map graph.*

2. *For every connected k-map graph $G = (V, E)$ with at least k vertices, there is a k-face sphere graph $\mathcal{H} = (V, E_{\mathcal{H}})$ whose common-face graph is a supergraph of G.*

PROOF. Statement 1 is obvious. To prove Statement 2, suppose that G is the half square of a bipartite planar graph $B = (V, P; E_B)$ such that each $p \in P$ has at most k neighbors in B. We may assume that B is simple and connected. A vertex $p \in P$ is *redundant* if its removal from B does not change the half square of B. We may further assume that B has no redundant vertex. Then, each $p \in P$ has at least two neighbors in B.

Consider an arbitrary embedding of B on the sphere and identify B with this embedding. We modify B as follows (see Figure 2.3 for an example): For every $p \in P$, add $d_B(p)$ new edges around p to connect the neighbors of p into a cycle C_p of length $d_B(p)$. Note that B may now have multiple edges. Obviously, each face F containing at least one $p \in P$ is a triangle. If B has a face of size at least 4, then we further modify B by (arbitrarily) triangulating every face of size at least 4. Now, every face of B is of size 2 or 3. We further modify B by removing all vertices in P and all edges incident to them. Note that the removal of each $p \in P$ (together with the edges incident to it) from B merges several faces into a single face F_p whose boundary is the cycle C_p. This implies that B is a k-face sphere graph. Moreover, for every edge $\{u, v\}$ of G, some $p \in P$ was adjacent to both u and v in B before the removal of p, and hence the face F_p contains both u and v after the removal of p. This implies that the common-face graph of B is a supergraph of G. □

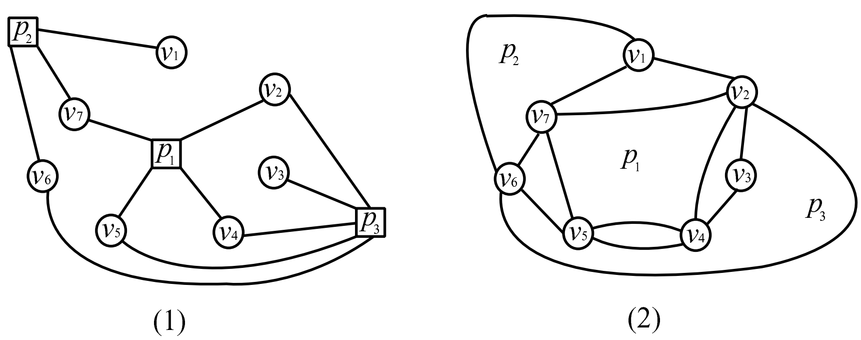


Figure 2.3: (1) A bipartite planar graph B. (2) The k-face sphere graph constructed from B.

3 Lower bounds

Theorem 3.1 *For every multiple n of 4 with $n \geq 8$, there is a 4-map graph with n vertices and $4n - 8$ edges.*

PROOF. Suppose that $n \geq 8$ is a multiple of 4. Construct a 4-face sphere graph $\mathcal{H}$ as follows. First, draw $\frac{n}{4}$ nested squares on the sphere so that the upper left and the lower right corners (respectively, the upper right and the lower left corners) of the squares are on the same line. Next, draw four lines L_{ul}, L_{ur}, L_{ll}, and L_{lr} to connect the upper left corners of the squares, the upper right corners of the squares, the lower left corners of the squares, and the lower right corners of the squares, respectively. Figure 3.1 illustrates the case where $n = 16$.

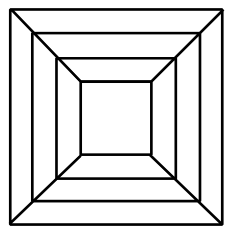


Figure 3.1: The case where $n = 16$.

As can be seen from Figure 3.1, each vertex u of $\mathcal{H}$ with $d_{\mathcal{H}}(u) = 4$ has degree 8 in the common-face graph of $\mathcal{H}$, while each vertex v of $\mathcal{H}$ with $d_{\mathcal{H}}(v) = 3$ has degree 6 in the common-face graph of $\mathcal{H}$. Since $\mathcal{H}$ has exactly 8 vertices v with $d_{\mathcal{H}}(v) = 3$ and exactly $n - 8$ vertices u with $d_{\mathcal{H}}(u) = 4$, the number of edges in the common-face graph of $\mathcal{H}$ is exactly $4n - 8$. □

Theorem 3.2 *For every integer $n \geq 20$ such that $n+10$ is a multiple of 15, there is a 5-map graph with n vertices and $5n - 10$ edges.*

PROOF. Suppose that $n \geq 20$ and $n + 10$ is a multiple of 15. Construct a 5-face sphere graph $\mathcal{H}$ as follows.

1. Draw $\ell = \frac{n+10}{15}$ nested pentagons $C_1, \ldots, C_\ell$ on the sphere.
2. For every $i \in \{1, \ldots, \ell - 1\}$, perform the following:
   (a) Draw a nested decagon D_i between C_i and C_{i+1}.
   (b) Let the vertices of D_i be $x_1, \ldots, x_{10}$ (appearing in D_i in this order). Let the vertices of C_i be $y_1, \ldots, y_5$ (appearing in C_i in this order). Let the vertices of C_{i+1} be $z_1, \ldots, z_5$ (appearing in C_{i+1} in this order). For each $j \in \{1, \ldots, 5\}$, connect x_{2j-1} to y_j by an edge and connect x_{2j} to z_j by an edge.

Figure 3.2 illustrates the case where $n = 50$.

As can be seen from Figure 3.2, each vertex u of $\mathcal{H}$ with $d_{\mathcal{H}}(u) = 4$ has degree 12 in the common-face graph of $\mathcal{H}$, while each vertex v of $\mathcal{H}$ with $d_{\mathcal{H}}(v) = 3$ has degree 9 in the common-face graph of $\mathcal{H}$. Now, a simple calculation shows that the number of edges in the common-face graph of $\mathcal{H}$ is exactly $5n - 10$. □

Theorem 3.3 *For every positive multiple k of 6, there are infinitely many integers n such that some k-map graph with n vertices has at least $(\frac{11}{12}k + \frac{1}{3})n$ edges.*

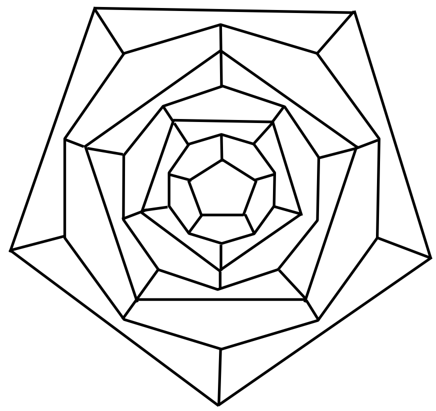
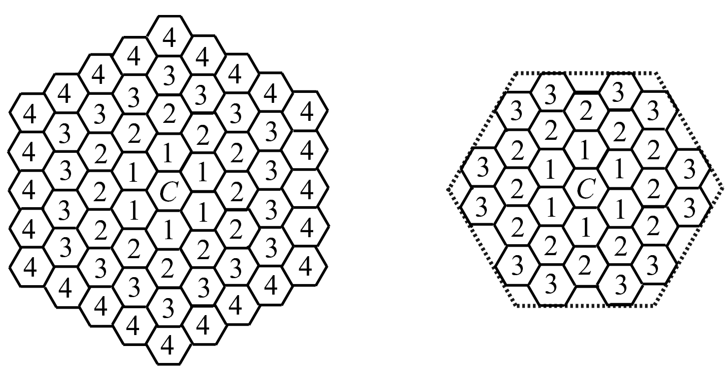


Figure 3.2: The case where $n = 50$.

Proof. Fix a positive multiple k of 6. Let C be an equilateral hexagon whose edges each are of length 1. Let r be a positive multiple of 3. We construct a plane graph $\mathcal{H}_1$ as follows. First, draw C in the plane (we call C the *central hexagon*). Then, draw six copies of C around C each of which shares exactly one edge with C (we say that these copies are *in layer* 1). Further draw twelve copies of C around the copies in layer 1 in such a way that each of the twelve new copies shares at least one edge with some copy in layer 1 (we say that these twelve new copies are *in layer* 2). Repeat in this way until a total of $\frac{2r}{3}$ layers of copies have been drawn. Note that for each $i \in \{1, \ldots, r\}$, the ith layer has $6i$ copies of C. Figure 3.3(1) illustrates the case where $r = 6$; the number inside each hexagon is the layer number of that hexagon.



(1) (2)

Figure 3.3: (1) $\mathcal{H}_1$ when $r = 6$. (2) $\mathcal{H}_2$ and C_r when $r = 6$. C_r is shown in broken lines.

From $\mathcal{H}_1$, we obtain another plane graph $\mathcal{H}_2$ as follows. Imagine that we extend the central hexagon C in all directions so that each of its edges becomes of length r. Let C_r be the resulting imaginary hexagon. Delete from $\mathcal{H}_1$ all vertices (together with the edges incident to them) that lie outside C_r. This completes the construction of $\mathcal{H}_2$. Figure 3.3(2) illustrates the case where $r = 6$. Obviously, the area bounded by C_r is r^2 times the area bounded by C, and $\mathcal{H}_1$ has exactly six internal faces (hexagons) F such that exactly one-third of the area of F is inside C_r. It is also easy to see that $\mathcal{H}_1$ has exactly $2r-6$ internal faces (hexagons) F such that exactly half the area of F is inside C_r. Thus, exactly $r^2 - \frac{6}{3} - \frac{2r-6}{2}$ internal faces (hexagons) of $\mathcal{H}_1$ remain to be internal faces of $\mathcal{H}_2$.

Let $\mathcal{H}'_2$ be a copy of $\mathcal{H}_2$. From $\mathcal{H}_2$ and $\mathcal{H}'_2$, we construct a 6-face sphere graph $\mathcal{H}_3$ as follows. Draw $\mathcal{H}_2$ on the upper sphere and draw $\mathcal{H}'_2$ on the lower sphere in such a way that each edge on the imaginary hexagon of $\mathcal{H}_2$ is identified with an edge on the imaginary hexagon of $\mathcal{H}'_2$ (more intuitively, their imaginary hexagons are identified). This completes the construction of $\mathcal{H}_3$. Note that there are exactly six faces of size 4 in $\mathcal{H}_3$. The other faces each are of size 6 and there are exactly $2(r^2-r+1)+(2r-6)$ of them according to our discussion in the last paragraph. Moreover, each vertex in $\mathcal{H}_3$ has degree 3. Thus, by Euler's formula, $\mathcal{H}_3$ has exactly $4r^2$ vertices and $6r^2$ edges.

Now, we construct a k-face sphere graph $\mathcal{H}$ from $\mathcal{H}_3$ as follows. First, for each face F of size 4, put $\frac{k}{3}$ isolated vertices in the interior of F. Then, for each edge e, put $\frac{k}{6}-1$ new vertices on e. This completes the construction of $\mathcal{H}$. Note that each face of $\mathcal{H}$ is of size k. Moreover, each nonisolated vertex in $\mathcal{H}$ is of degree 2 or 3, and each vertex of degree 3 is also a vertex in $\mathcal{H}_3$.

Let V_3 (respectively, V_2) be the set of those vertices v of degree 3 (respectively, 2) in $\mathcal{H}$. In the common-face graph of $\mathcal{H}$, each vertex in V_3 is of degree $\frac{5}{2}k-3$ and each vertex in V_2 is of degree $\frac{11}{6}k-2$. Obviously, each isolated vertex in $\mathcal{H}$ is of degree $k-1$ in the common-face graph of $\mathcal{H}$. On the other hand, $|V_3| = 4r^2$ and $|V_2| = 6r^2(\frac{k}{6}-1) = kr^2 - 6r^2$. Moreover, there are exactly $2k$ isolated vertices in $\mathcal{H}$. So, the total number n of vertices in $\mathcal{H}$ is $|V_3|+|V_2|+2k = kr^2-2r^2+2k$, and the degrees of the vertices in the common-face graph of $\mathcal{H}$ sum up to $(\frac{5}{2}k-3)|V_3| + (\frac{11}{6}k-2)|V_2| + 2k(k-1) = \frac{11}{6}k^2r^2 - 3kr^2 + 2k^2 - 2k$. Hence, the total number of edges in the common-face graph of $\mathcal{H}$ is $\frac{11}{12}k^2r^2 - \frac{3}{2}kr^2 + k^2 - k$ and hence is at least $(\frac{11}{12}k+\frac{1}{3})n$ for large enough r (say, $r \geq 2k$). This completes the proof. $\square$

4 A new upper bound

In this section, we show the following theorem:

Theorem 4.1 *Let ϵ be an arbitrary real number with $0 < \epsilon < \frac{3}{328}$, and let k be an integer not smaller than $\frac{140}{41\epsilon}$. Then, for every k-face sphere graph $\mathcal{H}$ with at least k vertices, the common-face graph of $\mathcal{H}$ has at most $(\frac{325}{328}+\epsilon)kn - 2k$ edges, where n is the number of vertices in $\mathcal{H}$.*

We prove Theorem 4.1 by contradiction. Assume that there is a k-face sphere graph with n ($\geq k$) vertices and more than $(\frac{325}{328}+\epsilon)kn - 2k$ edges. Call such a graph a *counterexample*. We choose $\mathcal{H}$ to be a counterexample such that $2(n+\#_f)^2 + \#_{cf} + 2\#_{df}$ is minimized among all counterexamples, where $\#_f$ is the number of faces in $\mathcal{H}$, $\#_{cf}$ is the number of critical faces in $\mathcal{H}$, and $\#_{df}$ is the number of dangerous faces in $\mathcal{H}$.

In the remainder of this section, we first prove some lemmas about the structure of $\mathcal{H}$. With these lemmas, we then prove that the common-face graph of $\mathcal{H}$ does not have so many edges; the proof is long and given in two subsections (Section 4.1 and 4.2).

Lemma 4.2 *The following statements hold:*

1. *No two strongly touching faces F_1 and F_2 in $\mathcal{H}$ can be merged without creating a face of size larger than k.*

2. *$\mathcal{H}$ has no small face F such that $V(F)$ contains an isolated vertex v.*

Proof. To prove Statement 1, assume that $\mathcal{H}$ has two strongly touching faces F_1 and F_2 which can be merged without creating a face of size larger than k. Then, modifying $\mathcal{H}$ by merging F_1 and F_2 yields a new k-face sphere graph $\mathcal{H}'$ with n vertices and $\#_f - 1$ faces. Moreover, the common-face graph of $\mathcal{H}'$ has at least as many edges as the common-face graph of $\mathcal{H}$. However, the existence of $\mathcal{H}'$ contradicts our choice of $\mathcal{H}$.

To prove Statement 2, assume that $\mathcal{H}$ has a small face F such that $V(F)$ contains an isolated vertex v. Obviously, modifying $\mathcal{H}$ by removing v yields a new k-face sphere graph $\mathcal{H}'$ with $n-1$ vertices and $\#_f$ faces. The common-face graph of $\mathcal{H}'$ has at most $\lceil \frac{k}{2} \rceil - 1$ fewer edges than the common-face graph of $\mathcal{H}$, and hence has more than $(\frac{325}{328} + \epsilon)kn - 2k - (\lceil \frac{k}{2} \rceil - 1) \geq (\frac{325}{328} + \epsilon)k(n-1) - 2k$ edges. However, the existence of $\mathcal{H}'$ contradicts our choice of $\mathcal{H}$. □

By Lemma 4.2, no two small faces of $\mathcal{H}$ can strongly touch, and hence each critical face strongly touches two large faces in $\mathcal{H}$. Moreover, each face of $\mathcal{H}$ has size at least 3.

Corollary 4.3 *No small face strongly touches exactly one face in $\mathcal{H}$.*

Proof. For a contradiction, assume that $\mathcal{H}$ has a small face F strongly touching exactly one face F'. Then, the boundary of F must be a cycle. Moreover, since F is small, $V(F)$ contains no isolated vertex by Statement 2 in Lemma 4.2. So, all vertices in $V(F)$ are on the boundary of F, and hence are also on the boundary of F'. However, this means that merging F and F' does not yield a face of size larger than k, contradicting Statement 1 in Lemma 4.2. □

Lemma 4.4 *Suppose that $\mathcal{H}$ has a dangerous face F. Let F_1 and F_2 be the two faces strongly touching F in $\mathcal{H}$. Then, we can shrink F and enlarge either one or both of F_1 and F_2 without modifying the other faces (than F, F_1, and F_2) of $\mathcal{H}$ to obtain a new k-face sphere graph $\mathcal{H}'$ satisfying the following five conditions:*

- *$\mathcal{H}'$ has n vertices and $\#_f$ faces.*

- *Both F_1 and F_2 have size k in $\mathcal{H}'$ (and hence F is not a dangerous face of $\mathcal{H}'$).*

- *Each dangerous face of $\mathcal{H}'$ is also a dangerous face of $\mathcal{H}$.*
- *Each critical face of $\mathcal{H}'$ other than F is also a critical face of $\mathcal{H}$.*
- *The common-face graph of $\mathcal{H}'$ has at least as many edges as the common-face graph of $\mathcal{H}$.*

Proof. Let $h = |(V(F) \cap V(F_1)) - V(F_2)|$ and $\ell = |(V(F) \cap V(F_2)) - V(F_1)|$. By Statement 1 in Lemma 4.2, merging F and F_1 yields a face of size larger than k; so $\ell + |V(F_1)| > k$ (and hence there is a nonnegative integer $j < \ell$ such that $j + |V(F_1)| = k$). Similarly, $h + |V(F_2)| > k$ (and hence there is a nonnegative integer $i < h$ such that $i + |V(F_2)| = k$). We next distinguish three cases as follows.

Case 1: F is a Type-1 critical face (cf. Figure 2.1(1)). Then, $V(F) \cap V(F_1) \cap V(F_2)$ consists of exactly two vertices w_1 and w_2. Moreover, the edges in $E(F) \cap E(F_1)$ form a path $P_1 = w_2, v_1, v_2, \ldots, v_h, w_1$ from w_2 to w_1. Similarly, the edges in $E(F) \cap E(F_2)$ form a path $P_2 = w_1, u_1, u_2, \ldots, u_\ell, w_2$ from w_1 to w_2. If the size of F_1 is less than k, then we delete edge $\{v_h, w_1\}$ and add edge $\{v_h, u_j\}$ (see the first arrow in Figure 4.1); this shrinks F and enlarges F_1 without modifying any other face so that the size of F_1 becomes k. If the size of F_2 is also less than k, then we further delete edge $\{u_\ell, w_2\}$ and add edge $\{u_\ell, v_i\}$ (see the second arrow in Figure 4.1); this shrinks F and enlarge F_2 without modifying any other face so that the size of F_2 becomes k. Let $\mathcal{H}'$ be the resulting k-face sphere graph. It is easy to verify that $\mathcal{H}'$ satisfies the five conditions in the lemma.

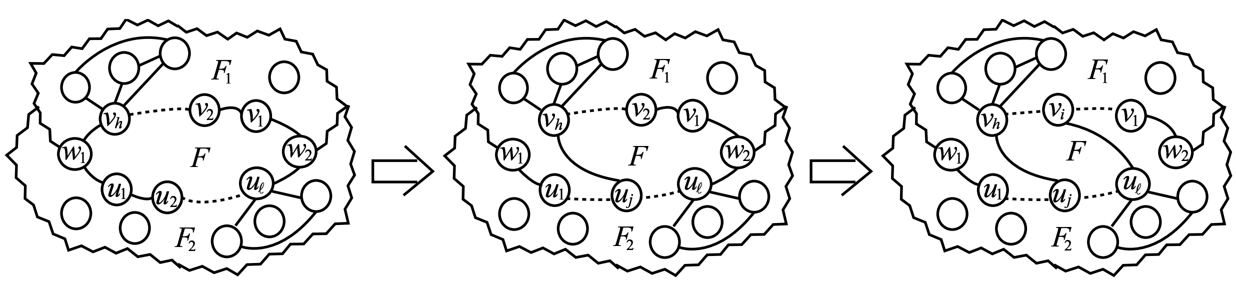


Figure 4.1: Shrinking a Type-1 critical face F to enlarge one face F_1 strongly touching F, and further shrinking F to enlarge the other face F_2 strongly touching F.

Case 2: F is a Type-2 critical face (cf. Figure 2.1(2)). Then, $V(F) \cap V(F_1) \cap V(F_2) = \emptyset$. Moreover, the edges in $E(F) \cap E(F_1)$ form a cycle $P_1 = v_1, v_2, \ldots, v_h, v_1$. Similarly, the edges in $E(F) \cap E(F_2)$ form a cycle $P_2 = u_1, u_2, \ldots, u_\ell, u_1$. If the size of F_1 is less than k, then we delete edge $\{v_1, v_h\}$ and add edges $\{v_1, u_j\}$ and $\{v_h, u_1\}$ (see the first arrow in Figure 4.2); this shrinks F and enlarges F_1 without modifying any other face so that the size of F_1 becomes k. If the size of F_2 is also less than k, then we further delete edge $\{u_\ell, u_1\}$ and add edge $\{u_\ell, v_{h-i+1}\}$ (see the second arrow in Figure 4.2); this shrinks F and enlarge F_2 without modifying any other face so that the size of F_2 becomes k. Let $\mathcal{H}'$ be the resulting k-face sphere graph. It is easy to verify that $\mathcal{H}'$ satisfies the five conditions in the lemma.

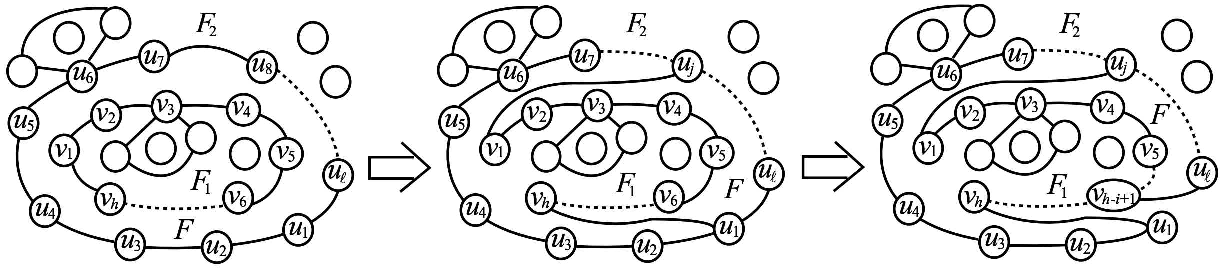


Figure 4.2: Shrinking a Type-2 critical face F to enlarge one face F_1 strongly touching F, and further shrinking F to enlarge the other face F_2 strongly touching F.

Case 3: F is a Type-3 critical face (cf. Figure 2.1(3)). Then, $V(F) \cap V(F_1) \cap V(F_2)$ consists of a single vertex w. Moreover, the edges in $E(F) \cap E(F_1)$ form a cycle $P_1 = w, v_1, v_2, \ldots, v_h, w$. Similarly, the edges in $E(F) \cap E(F_2)$ form a cycle $P_2 = w, u_1, u_2, \ldots, u_\ell, w$. If the size of F_1 is less than k, then we delete edge $\{v_1, w\}$ and add edge $\{v_1, u_j\}$ (see the first arrow in Figure 4.3); this shrinks F and enlarges F_1 without modifying any other face so that the size of F_1 becomes k. If the size of F_2 is also less than k, then we further delete edge $\{u_\ell, w\}$ and add edge $\{u_\ell, v_{h-i+1}\}$ (see the second arrow in Figure 4.3); this shrinks F and enlarge F_2 without modifying any other face so that the size of F_2 becomes k. Let $\mathcal{H}'$ be the resulting k-face sphere graph. It is easy to verify that $\mathcal{H}'$ satisfies the five conditions in the lemma.

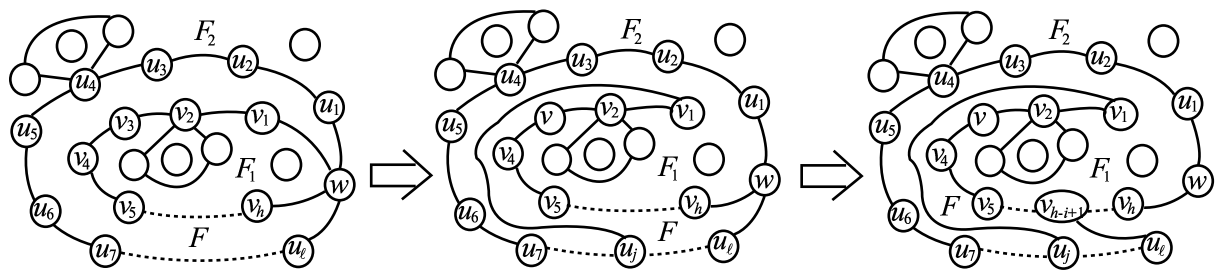


Figure 4.3: Shrinking a Type-3 critical face F to enlarge one face F_1 strongly touching F, and further shrinking F to enlarge the other face F_2 strongly touching F.

□

Corollary 4.5 *$\mathcal{H}$ has no dangerous face.*

Proof. Assume that $\mathcal{H}$ has a dangerous face F. Then, we can modify $\mathcal{H}$ to obtain a new k-face sphere graph $\mathcal{H}'$ satisfying the five conditions in Lemma 4.4. The existence of $\mathcal{H}'$ contradicts our choice of $\mathcal{H}$. □

Lemma 4.6 *Suppose that $\mathcal{H}$ has a critical vertex v_i. Let F be a critical face of $\mathcal{H}$ such that $v_i \in V(F)$ and v_i is on the boundary of exactly one (say, F_1) of the two faces F_1 and F_2 strongly touching F in $\mathcal{H}$. Then, one of the following statements holds:*

1. *We can modify $\mathcal{H}$ to obtain a new k-face sphere graph $\mathcal{H}'$ such that $\mathcal{H}'$ has n vertices and $\#_f - 1$ faces and the common-face graph of $\mathcal{H}'$ has at least as many edges as the common-face graph of $\mathcal{H}$.*

2. *We can modify F and F_1 without modifying the other faces (than F and F_1) to obtain a new k-face sphere graph $\mathcal{H}'$ satisfying the following six conditions:*

   - *$\mathcal{H}'$ has n vertices and $\#_f$ faces.*
   - *$\mathcal{H}'$ has no dangerous face.*
   - *F_1 still has size k in $\mathcal{H}'$ while the size of F in $\mathcal{H}'$ is larger than the size of F in $\mathcal{H}$ by 1.*
   - *F strongly touches F_1, F_2, and exactly one face $F_3 \notin \{F_1, F_2\}$ in $\mathcal{H}'$, (Comment: By this condition, F is not critical in $\mathcal{H}'$.)*
   - *Each critical face of $\mathcal{H}'$ is also a critical face of $\mathcal{H}$.*
   - *The common-face graph of $\mathcal{H}'$ has at least as many edges as the common-face graph of $\mathcal{H}$.*

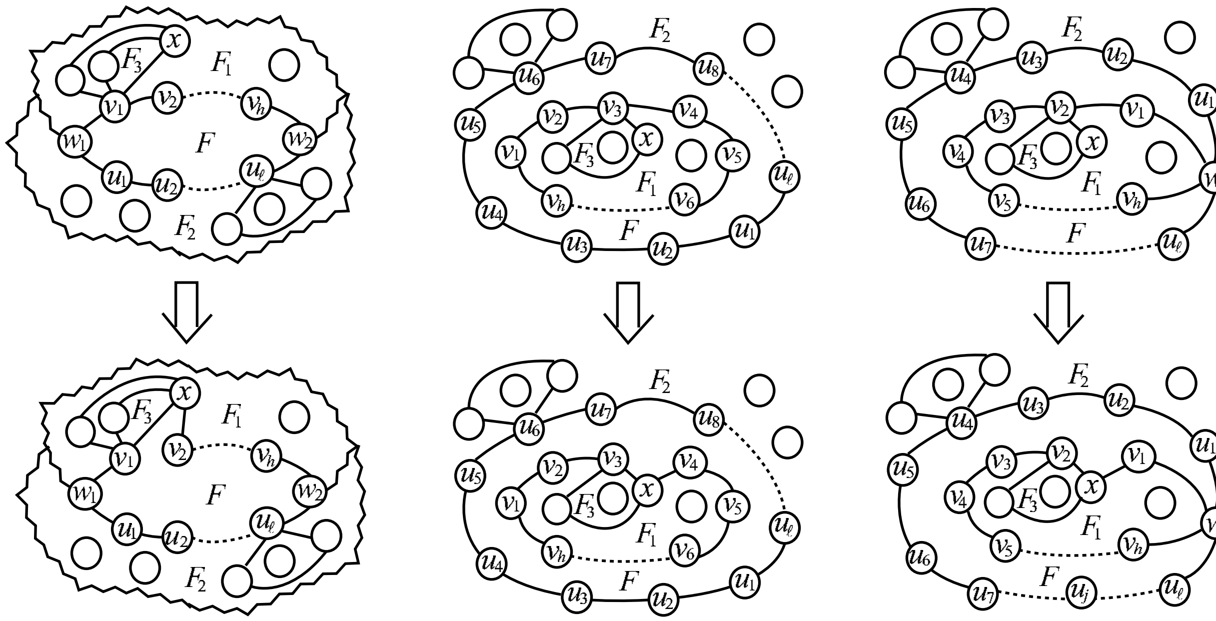
PROOF. Obviously, v_i has two neighbors v_j and x in $\mathcal{H}$ such that edges $\{v_i, v_j\}$ and $\{v_i, x\}$ appear around v_i consecutively in $\mathcal{H}$, edge $\{v_i, v_j\}$ is on the boundaries of F and F_1, and edge $\{v_i, x\}$ is on the boundaries of F_1 and a face $F_3 \notin \{F, F_1, F_2\}$. We shrink F_1 and enlarge F by deleting edge $\{v_i, v_j\}$ and adding edge $\{x, v_j\}$ (see Figure 4.4 for an example for each possible type of F); this does not modify the other faces than F and F_1. Let $\mathcal{H}'$ be the resulting k-face sphere graph. F_3 may be critical or even dangerous in $\mathcal{H}'$. If F_3 is not critical in $\mathcal{H}'$, then it is easy to verify that $\mathcal{H}'$ satisfies the six conditions in Statement 2 of the lemma. Otherwise, merging F_3 and F in $\mathcal{H}'$ does not yield a face of size larger than k because the size of F_3 (respectively, F) in $\mathcal{H}'$ is at most $\lceil \frac{k}{2} \rceil$ (respectively, $\lceil \frac{k}{2} \rceil + 1$) and hence merging them in $\mathcal{H}'$ yields a face of size at most $\lceil \frac{k}{2} \rceil + (\lceil \frac{k}{2} \rceil + 1) - 2 \le k$. So, if F_3 is critical in $\mathcal{H}'$, then further modifying $\mathcal{H}'$ by merging F_3 and F makes $\mathcal{H}'$ become a graph as described in Statement 1 in the lemma because merging two faces in $\mathcal{H}'$ decreases the number of faces in $\mathcal{H}'$ by 1 but does not decrease the number of edges in the common-face graph of $\mathcal{H}'$.

□

Corollary 4.7 *$\mathcal{H}$ has no critical vertex.*

PROOF. Assume that $\mathcal{H}$ has a critical vertex v_i. Then, we can modify $\mathcal{H}$ to obtain a new k-face sphere graph $\mathcal{H}'$ as in Lemma 4.6. The existence of $\mathcal{H}'$ contradicts our choice of $\mathcal{H}$. □

Let $k_2 = \lceil \frac{k}{2} \rceil$. Recall that $\mathcal{H}$ has no face of size at most 2. For each j with $3 \le j \le k$, let f_j be the number of faces of size j in $\mathcal{H}$. For each j with $3 \le j \le k_2$, let $f_{j,c}$ be the number of critical faces of size j in $\mathcal{H}$. Let n_0 be the number of isolated vertices in $\mathcal{H}$.

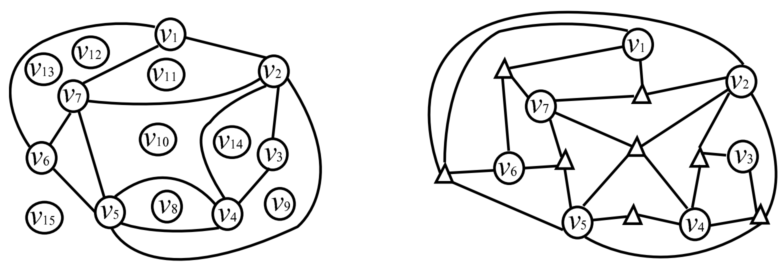


(1) (2) (3)

Figure 4.4: (1) Enlarging a Type-1 critical face F (where v_1 is critical). (2) Enlarging a Type-2 critical face F (where v_3 is critical). (3) Enlarging a Type-3 critical face F (where v_2 is critical).

Lemma 4.8 $\sum_{j=3}^{k}(j-2)f_j \leq 2n - n_0 - 4$.

Proof. Let $\mathcal{B}$ be the simple bipartite sphere graph obtained by modifying $\mathcal{H}$ as follows (see Figure 4.5 for an example). First, delete all isolated vertices of $\mathcal{H}$ and put a new vertex w_F inside each face F of $\mathcal{H}$. Next, connect w_F to each vertex in F. Finally, delete all original edges of $\mathcal{H}$.



(1) (2)

Figure 4.5: (1) A 5-face sphere graph $\mathcal{H}$. (2) The simple bipartite sphere graph $\mathcal{B}$ constructed from $\mathcal{H}$.

Obviously, $\mathcal{B}$ has exactly $\sum_{j=3}^{k} jf_j - n_0$ edges. Moreover, since $\mathcal{B}$ is a simple bipartite sphere graph with $n - n_0 + \sum_{j=3}^{k} f_j$ vertices, it can have at most $2(n - n_0 + \sum_{j=3}^{k} f_j) - 4$ edges. Thus, $\sum_{j=3}^{k}(j-2)f_j \leq 2n - n_0 - 4$. □

Fix a large constant β independent of k. The choice of β will become clear later.

4.1 The case when $\sum_{j=3}^{k_2}(j-2)f_{j,c} \geq \frac{n}{\beta}$

To bound the number of edges in the common-face graph of $\mathcal{H}$, our idea is to obtain a triangulated sphere graph $\mathcal{H}'$ by adding new edges to $\mathcal{H}$. Since the common-face graph of $\mathcal{H}'$ is itself (by Fact 2.1) and hence has at most $3n-6$ edges, it remains to bound the difference in the number of edges between the common-face graphs of $\mathcal{H}$ and $\mathcal{H}'$. We describe the details below.

Let $\mathcal{H}'$ be a copy of $\mathcal{H}$. For each face F of $\mathcal{H}$, let F' denote the face of $\mathcal{H}'$ corresponding to F. We modify $\mathcal{H}'$ by performing the following steps:

1. For each Type-1 critical face F of $\mathcal{H}$, if the two faces F_1 and F_2 strongly touching F in $\mathcal{H}$ satisfy $(V(F) \cap V(F_1)) - V(F_2) \neq \emptyset$ and $(V(F) \cap V(F_2)) - V(F_1) \neq \emptyset$, then triangulate F' in $\mathcal{H}'$ by adding $|V(F)| - 3$ new edges $\{x, y\}$ such that $x \in (V(F) \cap V(F_1)) - V(F_2)$ and $y \in (V(F) \cap V(F_2)) - V(F_1)$; otherwise, triangulate F' in $\mathcal{H}'$ arbitrarily. (Comment: There is no isolated vertex in F by Statement 2 in Lemma 4.2.)

2. For each Type-2 critical face F of $\mathcal{H}$, triangulate F' in $\mathcal{H}'$ by adding $|V(F)|$ new edges $\{x, y\}$ such that $x \in (V(F) \cap V(F_1)) - V(F_2)$ and $y \in (V(F) \cap V(F_2)) - V(F_1)$, where F_1 and F_2 are the two faces strongly touching F in $\mathcal{H}$. (Comment: There is no isolated vertex in F by Statement 2 in Lemma 4.2.)

3. For each Type-3 critical face F, triangulate F' in $\mathcal{H}'$ by adding $|V(F)| - 2$ new edges $\{x, y\}$ such that $x \in (V(F) \cap V(F_1)) - V(F_2)$ and $y \in (V(F) \cap V(F_2)) - V(F_1)$, where F_1 and F_2 are the two faces strongly touching F in $\mathcal{H}$. (Comment: There is no isolated vertex in F by Statement 2 in Lemma 4.2.)

4. For each noncritical face F, triangulate F' in $\mathcal{H}'$ by adding at least $(|V(F)| - 3) + i_F$ new edges, where i_F is the number of isolated vertices in F.

How many edges are in the common-face graph of $\mathcal{H}$ but are not in the common-face graph of $\mathcal{H}'$? To answer this question, we first prove the following lemma:

Lemma 4.9 *Let F be a face of $\mathcal{H}$. Then, the following statements hold:*

1. *If F is a Type-1 critical face of $\mathcal{H}$, then after Step 1 above, the common-face graph of $\mathcal{H}$ has at most $\frac{(|V(F)|-2)^2}{4} - (|V(F)| - 3)$ edges between vertices in $V(F)$ that are not edges in the common-face graph of $\mathcal{H}'$.*

2. *If F is a Type-2 critical face of $\mathcal{H}$, then after Step 2 above, the common-face graph of $\mathcal{H}$ has at most $\frac{|V(F)|^2}{4} - |V(F)|$ edges between vertices in $V(F)$ that are not edges in the common-face graph of $\mathcal{H}'$.*

3. *If F is a Type-3 critical face of $\mathcal{H}$, then after Step 3 above, the common-face graph of $\mathcal{H}$ has at most $\frac{(|V(F)|-1)^2}{4} - (|V(F)|-2)$ edges between vertices in $V(F)$ that are not edges in the common-face graph of $\mathcal{H}'$.*

4. *If F is a noncritical face of $\mathcal{H}$, then after Step 4 above, the common-face graph of $\mathcal{H}$ has at most $\frac{(|V(F)|-2)(|V(F)|-3)}{2}$ edges between vertices in $V(F)$ that are not edges in the common-face graph of $\mathcal{H}'$.*

PROOF. Let F_1 and F_2 be the two faces of $\mathcal{H}$ strongly touching F. Let $h_F = |(V(F) \cap V(F_1)) - V(F_2)|$ and $\ell_F = |(V(F) \cap V(F_2)) - V(F_1)|$. We prove the statements separately as follows.

(Statement 1) Let $\{u, v\} \subseteq V(F)$ be an edge in the common-face graph of $\mathcal{H}$. Since both F_1 and F_2 are large faces of $\mathcal{H}$ by Corollary 4.5, Step 1 has nothing to do with them. Thus, if $\{u, v\} \subseteq (V(F) \cap V(F_1)) - V(F_2)$ (respectively, $\{u, v\} \subseteq (V(F) \cap V(F_2)) - V(F_1)$), then even after Step 1, $\{u, v\}$ remains to be an edge between vertices of $V(F_1)$ (respectively, $V(F_2)$) in the common-face graph of $\mathcal{H}'$. So, if $h_F = 0$ or $\ell_F = 0$, then the statement trivially holds. Hence, suppose that $h_F > 0$ and $\ell_F > 0$. Now, if edge $\{u, v\}$ is newly added in Step 1, then it will remain to be an edge of the common-face graph of $\mathcal{H}'$ forever. Therefore, after Step 1, the common-face graph of $\mathcal{H}$ has at most $h_F \ell_F - (|V(F)|-3)$ edges between vertices in $V(F)$ that are not edges in the common-face graph of $\mathcal{H}'$. By elementary calculus, $h_F \ell_F - (|V(F)|-3) \leq \frac{(|V(F)|-2)^2}{4} - (|V(F)|-3)$. This completes the proof of the statement.

(Statement 2) Similar to the proof of Statement 1.

(Statement 3) Similar to the proof of Statement 1.

(Statement 4) Obviously, the common-face graph of $\mathcal{H}$ has at most $\frac{|V(F)|(|V(F)|-1)}{2}$ edges between vertices in $V(F)$. Moreover, there are at least $|V(F)| - i_F$ edges on the boundary of F in $\mathcal{H}$, where i_F is the number of isolated vertices in F. These edges will remain to be edges of the common-face graph of $\mathcal{H}'$ forever. In Step 4, at least $(|V(F)| - 3) + i_F$ new edges are added to $\mathcal{H}'$ and they will remain to be edges of the common-face graph of $\mathcal{H}'$ forever, too. Thus, the common-face graph of $\mathcal{H}$ has at most $\frac{|V(F)|(|V(F)|-1)}{2} - 2|V(F)| + 3 = \frac{(|V(F)|-2)(|V(F)|-3)}{2}$ edges between vertices in $V(F)$ that are not edges of the common-face graph of $\mathcal{H}'$. □

Since $\mathcal{H}'$ becomes a triangulated sphere graph, its common-face graph has at most $3n-6$ edges by Fact 2.1. Moreover, $\frac{(|V(F)|-1)^2}{4} - |V(F)| + 3 \geq \frac{|V(F)|^2}{4} - |V(F)|$ if and only if $|V(F)| \leq 6.5$. Thus, by Lemma 4.9, we have the following corollary.

Corollary 4.10 *The number of edges in the common-face graph of $\mathcal{H}$ is at most*

$$3n - 6 + \sum_{j=3}^{k} \frac{(j-2)(j-3)}{2}(f_j - f_{j,c}) + \sum_{j=3}^{6}(\frac{(j-1)^2}{4} - j + 3) f_{j,c} + \sum_{j=7}^{k_2}(\frac{j^2}{4} - j) f_{j,c}.$$

Now, we are ready to show a key lemma.

Lemma 4.11 *Suppose that* $\sum_{j=3}^{k_2}(j-2)f_{j,c} \geq \frac{n}{\beta}$. *Then, the common-face graph of* $\mathcal{H}$ *has at most* $(1-\frac{3}{8\beta}+\frac{9}{8\beta k})kn-2k$ *edges.*

PROOF. Since $k \geq \frac{140}{41\epsilon}$ and $\epsilon < \frac{3}{328}$, we have $k \geq 373$ and $k_2 \geq 186$. So, for every $j \in \{3,\ldots,6\}$, $\frac{(j-1)^2}{4} - j + 3 \leq \frac{k_2-2}{4}(j-2)$. Moreover, for every $j \in \{7,8,\ldots,k_2\}$, $\frac{j^2}{4} - j \leq \frac{k_2-2}{4}(j-2)$. Thus, by Corollary 4.10, the number of edges in the common-face graph of $\mathcal{H}$ is at most

$$\begin{aligned}
& 3n-6+\frac{k-3}{2}\sum_{j=3}^{k}(j-2)(f_j-f_{j,c})+\frac{k_2-2}{4}\sum_{j=3}^{k_2}(j-2)f_{j,c} \\
\leq\ & 3n-6+\frac{k-3}{2}(2n-4-\frac{n}{\beta})+\frac{k_2-2}{4}\cdot\frac{n}{\beta} \\
\leq\ & 3n-6+\frac{k-3}{2}(2n-4-\frac{n}{\beta})+\frac{k-3}{8}\cdot\frac{n}{\beta} \\
=\ & (1-\frac{3}{8\beta}+\frac{9}{8\beta k})kn-2k.
\end{aligned}$$

The first inequality above follows from Lemma 4.8, the assumption that $\sum_{j=3}^{k_2}(j-2)f_{j,c} \geq \frac{n}{\beta}$, and the observation that $\frac{k-3}{2} \geq \frac{k_2-2}{4}$. The second inequality above follows from the observation that $\frac{k-3}{2} \geq k_2 - 2$. □

4.2 The case when $\sum_{j=3}^{k_2}(j-2)f_{j,c} < \frac{n}{\beta}$

Let $\mathcal{H}_c$ be a copy of $\mathcal{H}$. We modify $\mathcal{H}_c$ by performing the following steps for each critical face F of $\mathcal{H}_c$:

1. Find a face F_1 strongly touching F such that the set $S_F = (V(F)\cap V(F_1)) - V(F_2)$ has size not smaller than $\ell_F = |(V(F)\cap V(F_2)) - V(F_1)|$, where F_2 is the other face strongly touching F in $\mathcal{H}_c$. (Comment: Merging F and F_1 yields a face of size exactly $|V(F_1)| + \ell_F$. Moreover, if we merge F and F_1 into a single face, then each vertex in S_F becomes an isolated vertex by Corollary 4.7.)

2. Merge F and F_1 into a single face F_3, and further delete exactly ℓ_F isolated vertices in S_F. (Comment: We claim that the size of F_3 is k after this step. To see this, first recall that before this step, the size of F_1 is k by Corollary 4.5. Now, by the comment on Step 1, the claim holds. By the claim, executing the two steps does not creat dangerous faces.)

By the above two steps, we have the following lemma immediately.

Lemma 4.12 $\mathcal{H}_c$ *has no critical face. Moreover, the two statements in Lemma 4.2 remain to hold even after replacing* $\mathcal{H}$ *by* $\mathcal{H}_c$.

Lemma 4.13 $\mathcal{H}_c$ *has no vertex of degree 1.*

PROOF. Initially, $\mathcal{H}_c$ has no vertex of degree 1 because it is a copy of $\mathcal{H}$ which is bridgeless. After merging two faces of $\mathcal{H}_c$, no vertex will become of degree 1. □

The next lemma will be used to show that the number of edges in the common-face graph of $\mathcal{H}_c$ is not so smaller than the number of edges in the common-face graph of $\mathcal{H}$.

Lemma 4.14 *The number of edges in the common-face graph of $\mathcal{H}$ is larger than the number of edges in the common-face graph of $\mathcal{H}_c$ by at most $\frac{9}{8}k(n - n_{\mathcal{R}})$, where $n_{\mathcal{R}}$ is the number of vertices in $\mathcal{H}_c$.*

PROOF. Let F be a critical face of $\mathcal{H}$. Suppose that we modify $\mathcal{H}$ by performing the above two steps for F only. Then, no matter which type F is, it is easy to see that in the common-face graph of $\mathcal{H}$ before the modification, each of the ℓ_F vertices deleted in Step 2 is adjacent to exactly k vertices not deleted in Step 2. Thus, after this modification, the number of edges in the common-face graph of $\mathcal{H}$ decreases by exactly $k\ell_F + \frac{\ell_F(\ell_F-1)}{2}$, which is not larger than $\frac{9}{8}k\ell_F$ because $\ell_F \le \lfloor\frac{|V(F)|}{2}\rfloor$ and $|V(F)| \le k_2$.

So, the number of edges in the common-face graph of $\mathcal{H}$ is larger than the number of edges in the common-face graph of $\mathcal{H}_c$ by at most $\frac{9}{8}k\sum_F \ell_F$, where the summation is taken over all critical faces F of $\mathcal{H}$. Note that $\sum_F \ell_F = n - n_{\mathcal{R}}$. Thus, the lemma holds. □

A crucial point is that the number of edges in the common-face graph of $\mathcal{H}_c$ is not so large. We will show this below.

A trouble with $\mathcal{H}_c$ is that it may have a large face which has very few edges on its boundary. So, instead of $\mathcal{H}_c$, we will work on a new k-face sphere graph $\mathcal{R}$ defined as follows. Initially, $\mathcal{R}$ is a copy of $\mathcal{H}_c$. We then modify this initial $\mathcal{R}$ by performing the following step for each large face F in it:

- If F has at least $\lceil\frac{k_2+2}{10}\rceil$ isolated vertices, then use new edges to connect the isolated vertices in F into a cycle C_F, in such a way that F is split into two faces F' and F'' one of which (say, F') has no vertex in its interior and has C_F as its boundary. (Comment: F'' contains the same vertices as F did, and hence is large. Moreover, F'' contains no isolated vertex. On the other hand, F' may be a small face, but its boundary must contain at least $\lceil\frac{k_2+2}{10}\rceil$ edges. Moreover, F' strongly touches F'' only.)

By the above comment, the common-face graph of $\mathcal{H}_c$ is the same as the common-face graph of $\mathcal{R}$. So, it suffices to work on $\mathcal{R}$ instead of $\mathcal{H}_c$.

We classify the small faces of $\mathcal{R}$ into two types as follows. For each small face F of $\mathcal{R}$, if F is also a small face of $\mathcal{H}_c$, then we say that F is *old*; otherwise, we say that F is *new*. The following lemma is clear from Corollary 4.3, Lemma 4.12, and the construction of $\mathcal{R}$:

Lemma 4.15 *The following statements hold:*

1. *No two small faces of $\mathcal{R}$ strongly touch in $\mathcal{R}$.*

2. *Each old small face of $\mathcal{R}$ strongly touches at least three large faces of $\mathcal{R}$.*

3. *Each new small face of $\mathcal{R}$ strongly touches exactly one large face of $\mathcal{R}$, and its boundary contains at least $\lceil \frac{k_2+2}{10} \rceil$ edges.*

4. *Each large face of $\mathcal{R}$ has at most $\lceil \frac{k_2+2}{10} \rceil - 1$ isolated vertices. Consequently, the boundary of each large face contains at least $(k_2+2) - \lceil \frac{k_2+2}{10} \rceil$ edges.*

Note that the number $n_{\mathcal{R}}$ in Lemma 4.14 is also the number of vertices in $\mathcal{R}$. We need several more notations. Let

- $m_{\mathcal{R}}$ be the number of edges in $\mathcal{R}$,
- λ_j be the number of faces of size j in $\mathcal{R}$ for each $j \geq 3$,
- λ_{big} be the number of large faces in $\mathcal{R}$,
- λ_{new} be the number of new small faces in $\mathcal{R}$,
- λ_{old} be the number of old small faces in $\mathcal{R}$,
- $\mathcal{R}^*$ be the dual graph of $\mathcal{R}$,
- S^* be the underlying simple graph of $\mathcal{R}^*$ (i.e., S^* is the simple graph obtained from G by deleting multiple edges), and
- m_{S^*} be the number of edges in S^*.

Lemma 4.16 *$m_{\mathcal{R}} \geq \frac{k}{40}(9\lambda_{\text{big}} + \lambda_{\text{new}})$ and $m_{S^*} \leq 9\lambda_{\text{big}} + \lambda_{\text{new}}$.*

PROOF. By Statement 4 in Lemma 4.15, the boundary of each large face of $\mathcal{R}$ contains at least $\frac{9k}{20}$ edges. Moreover, by Statement 3 in Lemma 4.15, the boundary of each new small face of $\mathcal{R}$ contains at least $\frac{k}{20}$ edges. Further note that each edge appears on the boundary of exactly two faces. Thus, the first inequality in the lemma holds.

To prove the second inequality, we first claim that $3\lambda_{\text{old}} \leq 2(\lambda_{\text{big}} + \lambda_{\text{old}}) - 4$. To see this claim, it suffices to consider a simple bipartite sphere graph $\mathcal{B} = (V_{\mathcal{B},l}, V_{\mathcal{B},o}; E_{\mathcal{B}})$, where $V_{\mathcal{B},l}$ is the set of all large faces, $V_{\mathcal{B},o}$ is the set of all old small faces of $\mathcal{R}$, and $E_{\mathcal{B}}$ consists of all $\{F_1, F_2\}$ such that $F_1 \in V_{\mathcal{B},l}$, $F_2 \in V_{\mathcal{B},o}$, and they strongly touch in $\mathcal{R}$. Since $\mathcal{B}$ is a simple bipartite sphere graph, it has at most $2(\lambda_{\text{big}} + \lambda_{\text{old}}) - 4$ edges. On the other hand, by Statement 2 in Lemma 4.15, $\mathcal{B}$ has at least $3\lambda_{\text{old}}$ edges. Thus, the claim holds. By this claim, $\lambda_{\text{old}} \leq 2\lambda_{\text{big}} - 4$.

Since S^* is a simple sphere graph, the subgraph of S^* induced by the set of those vertices that correspond to the large faces and the old small faces of $\mathcal{R}$ has at most $3(\lambda_{\text{big}} + \lambda_{\text{old}}) - 6$ edges.

Besides these edges, S^* has at most λ_{new} other edges by Statements 1 and 3 in Lemma 4.15. Thus, $m_{S^*} \le 3(\lambda_{\mathrm{big}} + \lambda_{\mathrm{old}}) + \lambda_{\mathrm{new}} - 6$. Recall that $\lambda_{\mathrm{old}} \le 2\lambda_{\mathrm{big}} - 4$. Therefore, $m_{S^*} \le 9\lambda_{\mathrm{big}} + \lambda_{\mathrm{new}} - 18$. This completes the proof. □

By Lemma 4.16, the average number of multiple edges between a pair of adjacent vertices in $\mathcal{R}^*$ is at least $\frac{k}{40}$, which is large if k is large. Moreover, if there are many multiple edges between a pair $\{F_1, F_2\}$ of adjacent vertices in $\mathcal{R}^*$, then the two faces F_1 and F_2 of $\mathcal{R}$ must share many vertices. This is a key for us to show that the common-face graph of $\mathcal{R}$ does not have so many edges. We will clarify this below.

Let $\mathcal{R}'$ be a copy of $\mathcal{R}$. For each face F of $\mathcal{R}$, let F' be the face of $\mathcal{R}'$ corresponding to F. We modify $\mathcal{R}'$ by triangulating each face of $\mathcal{R}'$. We want to estimate how many edges are in the common-face graph of $\mathcal{R}$ but are not in the common-face graph of $\mathcal{R}'$.

Lemma 4.17 *If we modify $\mathcal{R}$ by triangulating exactly one face F, then the number of edges in the common-face graph of $\mathcal{R}$ decreases by at most $\frac{(|V(f)|-2)(|V(f)|-3)}{2}$. Consequently, the number of edges in the common-face graph of $\mathcal{R}$ is at most $3n_{\mathcal{R}} - 6 + \sum_{j=3}^{k} \frac{(j-2)(j-3)}{2}\lambda_j$.*

PROOF. The proof is similar to that of Statement 4 in Lemma 4.9. Before triangulating F, the common-face graph of $\mathcal{R}$ has at most $\frac{|V(F)|(|V(F)|-1)}{2}$ edges between vertices in $V(F)$. After modifying $\mathcal{R}$ by triangulating F, $\mathcal{R}$ has at least $2|V(F)| - 3$ edges between vertices in $V(F)$ and these edges are still in the common-face graph of $\mathcal{R}$. Thus, triangulating F causes the common-face graph of $\mathcal{R}$ to lose at most $\frac{(|V(f)|-2)(|V(f)|-3)}{2}$ edges.

Now, since $\mathcal{R}'$ is obtained by triangulating all faces of $\mathcal{R}$ one after another, the number of edges in the common-face graph of $\mathcal{R}$ but not in the common-face graph of $\mathcal{R}'$ is at most $\sum_{j=3}^{k} \frac{(j-2)(j-3)}{2}\lambda_j$. Moreover, the number of edges in the common-face graph of $\mathcal{R}'$ is at most $3n_{\mathcal{R}} - 6$. Thus, the number of edges in the common-face graph of $\mathcal{R}$ is at most $3n_{\mathcal{R}} - 6 + \sum_{j=3}^{k} \frac{(j-2)(j-3)}{2}\lambda_j$. □

The above estimate of the number of edges in the common-face graph of $\mathcal{R}$ is too pessimistic. The next key lemma gives a better estimate.

Lemma 4.18 *Let F_1 and F_2 be two strongly touching faces of $\mathcal{R}$. Suppose that there are $r_{1,2}$ multiple edges between F_1 and F_2 in $\mathcal{R}^*$. Then, the following hold:*

1. *There is a set $X_{1,2}$ of at least $r_{1,2}$ vertices shared by the boundaries of F_1 and F_2 such that no two vertices of $X_{1,2}$ are shared by the boundaries of two faces F_3 and F_4 of $\mathcal{R}$ with $\{F_1, F_2\} \ne \{F_3, F_4\}$.*

2. *If we modify $\mathcal{R}$ by triangulating F_1 and F_2 only, then the number of edges in the common-face graph of $\mathcal{R}$ decreases by at most $\sum_{i=1}^{2} \frac{(|V(F_i)|-2)(|V(F_i)|-3)}{2} - \frac{r_{1,2}^2 - 7r_{1,2} + 12}{2}$.*

3. *The common-face graph of $\mathcal{R}$ has at most $3n_{\mathcal{R}} - 6 + \sum_{j=3}^{k} \frac{(j-2)(j-3)}{2}\lambda_j - \frac{k-280}{80} m_{\mathcal{R}}$ edges.*

Proof. We prove the three statements in turn as follows.

(Statement 1) Consider the subgraph of $\mathcal{R}$ whose edges are the edges shared by the boundaries of F_1 and F_2 and whose vertices are the endpoints of these edges. The subgraph is either a cycle of length $r_{1,2}$ or a collection of vertex-disjoint paths each of which has length at least 1. In the former case, we set $X_{1,2}$ to be the set of vertices on the cycle. In the latter case, we traverse the vertex-disjoint paths clockwise; during the traversal of each path, we let $X_{1,2}$ include all vertices except the last one on the path. Obviously, in both cases, $X_{1,2}$ is as required.

(Statement 2) By Lemma 4.17, triangulating F_1 (respectively, F_2) only causes the common-face graph of $\mathcal{R}$ to lose at most $\frac{(|V(F_1)|-2)(|V(F_1)|-3)}{2}$ (respectively, $\frac{(|V(F_2)|-2)(|V(F_2)|-3)}{2}$ edges. For each $i \in \{1,2\}$, call the quantity $\frac{(|V(F_i)|-2)(|V(F_i)|-3)}{2}$ the *pessimistic loss* of F_i.

Let $\mathcal{K}$ be a k-face sphere graph obtained from $\mathcal{R}$ by triangulating F_1 and F_2 only. Let E_1 (respectively, E_2) be the set of edges in $\mathcal{K}$ between vertices in $V(F_1)$ (respectively, $V(F_2)$). Let u and v be two vertices in $X_{1,2}$. By the proof of Lemma 4.17, we have the following observations:

- Suppose that $E_1 \cup E_2$ contains an edge between u and v. Then, for each $i \in \{1,2\}$, the edge $\{u,v\}$ is not counted in the pessimistic loss of F_i.
- Suppose that $E_1 \cup E_2$ contains no edge between u and v. Then, for each $i \in \{1,2\}$, the edge $\{u,v\}$ is counted in the pessimistic loss of F_i.

Also note that there are at most $3|X_{1,2}|-6$ (unordered) pairs $\{u,v\} \subseteq X_{1,2}$ such that $E_1 \cup E_2$ contains an edge between u and v. Thus, by the above observations, at least $\frac{|X_{1,2}|(|X_{1,2}|-1)}{2} - (3|X_{1,2}|-6)$ edges are counted in both the pessimistic loss of F_1 and the pessimistic loss of F_2. Hence, triangulating F_1 and F_2 only causes the common-face graph of $\mathcal{R}$ to lose at most $\sum_{i=1}^{2} \frac{(|V(F_i)|-2)(|V(F_i)|-3)}{2} - \frac{r_{1,2}^2 - 7r_{1,2}+12}{2}$ edges.

(Statement 3) Let $\mathcal{P}$ be the set of all (unordered) pairs $\{F_i, F_j\}$ such that F_i and F_j are strongly touching faces of $\mathcal{R}$. For each pair $\{F_i, F_j\} \in \mathcal{P}$, let $r_{i,j}$ be the number of edges shared by the boundaries of F_i and F_j in $\mathcal{R}$, and let $X_{i,j}$ be a set of at least $r_{i,j}$ vertices shared by the boundaries of F_i and F_j such that no two vertices of $X_{i,j}$ are shared by the boundaries of two faces $F_{i'}$ and $F_{j'}$ of $\mathcal{R}$ with $\{F_i, F_j\} \neq \{F_{i'}, F_{j'}\}$. $X_{i,j}$ exists because of Statement 1.

By Statement 2, the pessimistic estimate (namely, $\frac{(|V(F_i)|-2)(|V(F_i)|-3)}{2} + \frac{(|V(F_j)|-2)(|V(F_j)|-3)}{2}$) of loss in the number of edges in the common-face graph of $\mathcal{R}$ after triangulating F_i and F_j only, overcounts at least $\frac{r_{i,j}^2 - 7r_{i,j}+12}{2}$ edges. We associate these overcounted edges with $X_{i,j}$.

Let $\{F_i, F_j\}$ and $\{F_{i'}, F_{j'}\}$ be two distinct pairs in $\mathcal{P}$. By our choices of $X_{i,j}$ and $X_{i',j'}$, the set of the overcounted edges associated with $X_{i,j}$ does not intersect the set of the overcounted edges associated with $X_{i',j'}$. Thus, by Lemma 4.17, the number of edges in the common-face graph of $\mathcal{R}$ is at most $3n_{\mathcal{R}} - 6 + \sum_{j=3}^{k} \frac{(j-2)(j-3)}{2}\lambda_j - \sum_{\{F_i,F_j\}\in\mathcal{P}} \frac{r_{i,j}^2 - 7r_{i,j}+12}{2}$. Note that $|\mathcal{P}| = m_{S^*}$ and $\sum_{\{F_i,F_j\}\in\mathcal{P}} r_{i,j} = m_{\mathcal{R}}$. Thus, by simple calculus, the number of edges in the common-face graph of $\mathcal{R}$ is at most $3n_{\mathcal{R}} - 6 + \sum_{j=3}^{k} \frac{(j-2)(j-3)}{2}\lambda_j - \frac{m_{\mathcal{R}}^2}{2m_{S^*}} + \frac{7}{2}m_{\mathcal{R}} - 6m_{S^*}$, and hence is at most $3n_{\mathcal{R}} - 6 + \sum_{j=3}^{k} \frac{(j-2)(j-3)}{2}\lambda_j - \frac{k-280}{80}m_{\mathcal{R}}$ by Lemma 4.16. □

Corollary 4.19 *The common-face graph of $\mathcal{R}$ has at most $(\frac{79}{80}k + \frac{7}{2})n_{\mathcal{R}} - 2k$ edges.*

Proof. Let $i_{\mathcal{R}}$ be the number of isolated vertices in $\mathcal{R}$. Similarly to Lemma 4.8, we can show that $\sum_{j=3}^{k}(j-2)\lambda_j \leq 2n_{\mathcal{R}} - i_{\mathcal{R}} - 4$. Moreover, since $\mathcal{R}$ has no vertex of degree 1 (by Lemma 4.13 and the construction of $\mathcal{R}$ from $\mathcal{H}_c$), $m_{\mathcal{R}} \geq n_{\mathcal{R}} - i_{\mathcal{R}}$. Thus, by Statement 3 in Lemma 4.18, the number of edges in the common-face graph of $\mathcal{R}$ is at most $3n_{\mathcal{R}} - 6 + \frac{k-3}{2}(2n_{\mathcal{R}} - i_{\mathcal{R}} - 4) - \frac{k-280}{80}(n_{\mathcal{R}} - i_{\mathcal{R}}) \leq (\frac{79}{80}k + \frac{7}{2})n_{\mathcal{R}} - 2k$. □

Lemma 4.20 *Suppose that $\sum_{j=3}^{k_2}(j-2)\lambda_{j,c} < \frac{n}{\beta}$. Then, the number of edges in the common-face graph of $\mathcal{H}$ is less than $(\frac{79}{80} + \frac{11}{80\beta} + \frac{7(\beta-1)}{2\beta k})kn - 2k$.*

Proof. When constructing $\mathcal{R}$ from $\mathcal{H}$, we deleted at most $\lfloor\frac{|V(F)|}{2}\rfloor$ vertices from each critical face F of $\mathcal{H}$. Thus, $n - n_{\mathcal{R}} \leq \sum_F \lfloor\frac{|V(F)|}{2}\rfloor$, where the summation is taken over all critical faces F of $\mathcal{H}$. Since $|V(F)| \geq 3$, $n - n_{\mathcal{R}} \leq \sum_F(|V(F)| - 2) = \sum_{j=3}^{k_2}(j-2)\lambda_{j,c}$. So, $n_{\mathcal{R}} > \frac{\beta-1}{\beta}n$.

Now, by Lemma 4.14 and Corollary 4.19, the number of edges in the common-face graph of $\mathcal{H}$ is at most $\frac{9}{8}k(n - n_{\mathcal{R}}) + (\frac{79}{80}k + \frac{7}{2})n_{\mathcal{R}} - 2k = \frac{9}{8}kn - (\frac{11}{80}k - \frac{7}{2})n_{\mathcal{R}} - 2k < (\frac{79}{80} + \frac{11}{80\beta} + \frac{7(\beta-1)}{2\beta k})kn - 2k$, because $k \geq 373$. □

Setting $\beta = 41$ and combining Lemmas 4.11 and 4.20, we finally have that the common-face graph of $\mathcal{H}$ has at most $(\frac{325}{328} + \epsilon)kn - 2k$ edges, a contradiction against our choice of $\mathcal{H}$. This completes the proof of Theorem 4.1.

By Theorem 4.1 and Lemma 2.2, we have the following corollary:

Corollary 4.21 *Let ϵ be an arbitrary real number with $0 < \epsilon < \frac{3}{328}$, and let k be an integer not smaller than $\frac{140}{41\epsilon}$. Then, every k-map graph G with at least k vertices has at most $(\frac{325}{328} + \epsilon)kn - 2k$ edges, where n is the number of vertices in G.*

5 Open Problems

For integers $k \geq 6$, it would very interesting to find a tight bound on the edge number of k-map graphs. An easier but still interesting question is to ask if one can improve the bounds proved in this paper.

It is known that 4-map graphs are 6-colorable [4] and there is a linear-time algorithm for 7-coloring 4-map graphs [8]. It would be interesting to design efficient algorithms for coloring k-map graphs with $k \geq 5$ using as few colors as possible. For this purpose, (almost) tight bounds on the edge number of a k-map graph seem necessary.

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