A Linear-Time Algorithm for 7-Coloring 1-Planar Graphs

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Abstract

A graph G is 1-planar if it can be embedded in the plane in such a way that each edge crosses at most one other edge. Borodin showed that 1-planar graphs are 6-colorable, but his proof does not lead to an efficient algorithm. This paper presents a linear-time algorithm for 7-coloring 1-planar graphs. The main difficulty in the design of our algorithm comes from the fact that the class of 1-planar graphs is not closed under the operation of edge contraction. This difficulty is overcome by a structure lemma that may find useful in other problems on 1-planar graphs. This paper also shows that it is NP-complete to decide whether a given 1-planar graph is 4-colorable. The complexity of the problem of deciding whether a given 1-planar graph is 5-colorable is still unknown.

Key words. Planar graphs, plane embeddings, 1-planar graphs, 1-plane embeddings, graph algorithms, vertex coloring, NP-completeness.

AMS subject classifications. 05C85, 68R05, 68R10, 68W40.

Abbreviated title. 7-Coloring 1-Planar Graphs.

1 Introduction

The problem of coloring the vertices of a graph using few colors has been a central problem in graph theory. It has also been intensively studied in algorithm theory due to its applications in many practical fields such as scheduling, resource allocation, and VLSI design. Of special interest is the case where the graph is planar. Appel and Haken [1, 2] showed that every planar graph is 4-colorable, but their proof does not lead to an efficient algorithm. Since then, a number of linear-time algorithms for 5-coloring planar graphs have appeared [9, 14, 11, 19, 10].

An interesting generalization of planar graphs is the class of 1-planar graphs. The problem of coloring the vertices of a 1-planar graph using few colors has also attracted very much attention [16, 15, 3, 4, 5]. Indeed, the problem has been formulated in another different way: It is equivalent to the problem of coloring the vertices of a plane graph so that the boundary vertices of every face of size at most 4 receive different colors [15]. Ringel [16] proved that every 1-planar graph is 7-colorable and conjectured that every 1-planar graph is 6-colorable. Ringel [16] and Archdeacon [3] confirmed the conjecture for two special cases. Borodin [4] settled the conjecture in the affirmative with a lengthy proof. He [5] later came up with a relatively shorter proof. However, his proof does not lead to an efficient algorithm for 6-coloring 1-planar graphs.

Chen, Grigni, and Papadimitriou [7] studied a modified notion of planarity, in which two nations of a (political) map are considered adjacent when they share any *point* of their boundaries (not necessarily an *edge*, as planarity requires). Such adjacencies define a *map graph* (see [8] for a comprehensive survey of known results on map graphs). The map graph is called a k-*map graph* if no more than k nations on the map meet at a point. As observed in [7], the adjacency graph of the

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United States is nonplanar but is a 4-map graph (see Figure 1). Obviously, every 4-map graph is 1-planar. In Section 4, we will observe that every 1-planar graph can be modified to a 4-map graph by adding some edges (see Corollary 4.3 below). By these facts, the problem of coloring 1-planar graphs is essentially equivalent to the problem of coloring 4-map graphs.

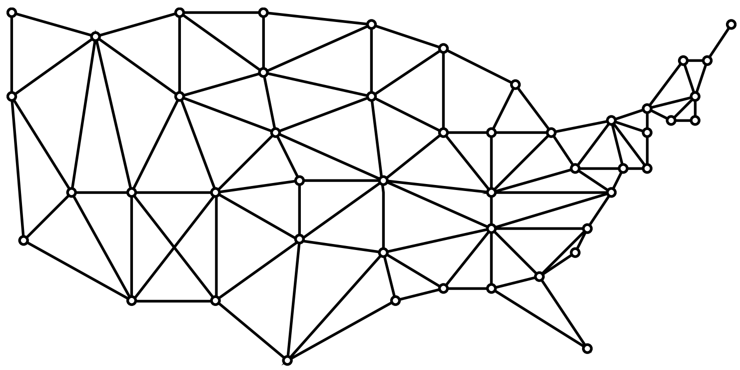


Figure 1: The USA map graph.

Recall that in the case of planar graphs, an efficient 4-coloring algorithm seems to be difficult to design and hence it is of interest to look for an efficient 5-coloring algorithm. Similarly, in the case of 1-planar graphs, an efficient 6-coloring algorithm seems to be difficult to design and hence it is of interest to look for an efficient 7-coloring algorithm. In this paper, we present the first linear-time algorithm for 7-coloring 1-planar graphs. Our algorithm is much more complicated than all 5-coloring algorithms for planar graphs. The main reason is that unlike planar graphs, the class of 1-planar graphs is not closed under the operation of *edge contraction* (recall that *contracting an edge* $\{u, v\}$ in a graph G is made by replacing u and v by a single new vertex z and adding an edge between z and each original neighbor of u and/or v). Figure 2 shows a 1-planar graph G; a simple inspection convinces us that contracting the edge $\{2, 4\}$ of G results in a graph that is not 1-planar. It is worth noting that many coloring algorithms (e.g., those for planar graphs) are crucially based on the property that the class of their input graphs is closed under the operation of edge contraction. In the case of 1-planar graphs, this property is not available and it becomes difficult to find suitable vertices to merge so that the resulting graph is still 1-planar. We overcome this difficulty with a structure lemma which essentially says that every 1-planar graph either has a constant fraction of vertices of degree at most 7, or has a constant fraction of vertices each of which is of degree 8 and has at least 5 neighbors of degree at most 8. We believe that this lemma will find useful in the design of algorithms for other problems on 1-planar graphs.

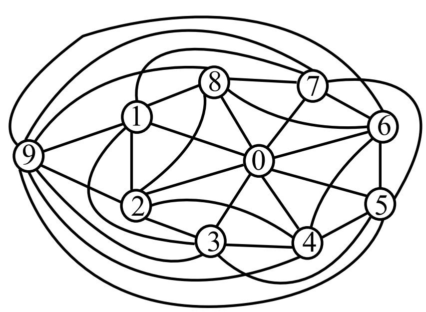


Figure 2: A 1-planar graph.

Since planar graphs are special 1-planar graphs and it is NP-complete to decide whether a given planar graph is 3-colorable [12], it is also NP-complete to decide whether a given 1-planar graph is 3-colorable. This paper shows that it is NP-complete to decide whether a given 1-planar graph is 4-colorable. The problem of deciding whether a given 1-planar graph is 5-colorable remains open.

The remainder of this paper is organized as follows. Section 2 describes several basic definitions and proves several elementary facts. Section 4 describes the linear-time algorithm. Section 5 details a proof for a key theorem needed for the algorithm. Section 6 proves the NP-hardness of the problem of deciding whether a given 1-planar graph is 4-colorable.

2 Preliminaries

Throughout this paper, a graph is always simple (i.e., has neither multiple edges nor self-loops) unless stated explicitly otherwise.

Let $G = (V, E)$ be a graph. The *neighborhood* of a vertex v in G, denoted $N_G(v)$, is the set of vertices in G adjacent to v; $d_G(v) = |N_G(v)|$ is the *degree* of v in G. For $U \subseteq V$, let $N_G(U) = \cup_{u \in U} N_G(u)$. For $U \subseteq V$, the *subgraph of G induced by U* is the graph (U, F) with $F = \{\{u, v\} \in E : u, v \in U\}$ and is denoted by $G[U]$. For $U \subseteq V$, we denote by $G - U$ the subgraph induced by $V - U$. If $u \in V$, we write $G - u$ instead of $G - \{u\}$. A *cut set* of G is a subset U of V such that $G - U$ is disconnected. A *k-cut set* is a cut set consisting of k vertices. G is *k-connected* if it has at least k vertices but has no i-cut set with $i \leq k - 1$. An *independent set* in G is a set of pairwise nonadjacent vertices in G. A *maximal independent set* in G is an independent set in G that is not a proper subset of another independent set in G.

A *1-plane embedding* of G is an embedding of G in the plane in such a way that each edge crosses at most one other edge. G has a 1-plane embedding only when G is a 1-planar graph. An *edge-crossing list* of G is a list L of disjoint (unordered) pairs of edges of G such that G has a 1-plane embedding in which the two edges in each pair in L cross while no two other edges of G cross.

For a sequence $\langle u_1, \ldots, u_k \rangle$ of two or more distinct pairwise nonadjacent vertices in G, *merging* $\langle u_1, \ldots, u_k \rangle$ is the operation of modifying G by adding an edge between u_k and every vertex in $\cup_{1 \leq i \leq k-1} N_G(u_i) - N_G(u_k)$ and further removing vertices $u_1, \ldots, u_{k-1}$. Note that the sequence is ordered and u_k is the last vertex in the sequence.

Let k be a natural number. A *k-coloring* of G is a coloring of the vertices of G with at most k colors such that no two adjacent vertices get the same color. The *color classes* of a coloring $\mathcal{C}$ of the vertices of G are the sets V_1, V_2, ..., V_k, where k is the number of colors used by $\mathcal{C}$ and V_i, $1 \leq i \leq k$, is the set of all vertices with the ith color.

3 Simple Reductions

In this section, we show how to reduce the problem to its 3-connected case. Throughout this section, G denotes the input 1-planar graph.

Since it is still unknown whether 1-planar graphs can be recognized in polynomial time, we assume that G is given by its adjacency list together with an edge-crossing list L of G. We may further assume that for each pair $(e, e') \in L$, e and e' share no endpoint (otherwise, the pair (e, e') can be removed from L, and L remains an edge-crossing list of G after the removal).

In the initialization step of the algorithm, we augment G as follows: For each pair $(e, e') \in L$, each endpoint u of e, and each endpoint v of e', if (u, v) is not an edge of G, then add a new edge between u and v in G. Obviously, this augmentation can be done in linear time. Moreover, after the augmentation, it is clear that (1) G remains 1-planar and (2) for each pair $(e, e') \in L$, the endpoints of e and e' induce a clique of size 4 in G.

We may assume that G is connected, since otherwise the problem of 7-coloring G is easily reduced to the problems of 7-coloring the connected components of G. A *block* of G is a maximal 2-connected subgraph of G. The following fact is widely known.

Fact 3.1 *Suppose that G is connected but not 2-connected. Let B be the set of all blocks of G. Let C be the class of all 1-cut sets of G. Consider the bipartite graph $T = (B \cup C, E_T)$ where E_T*

consists of all edges $\{b, c\}$ such that $b \in B$, $c \in C$, and the vertex in c is a vertex in B. Then, T is a tree.

Corollary 3.2 *Suppose that G is 1-planar and connected but not 2-connected. Then, given the blocks of G and a 7-coloring of each block of G, we can compute a 7-coloring of G in linear time.*

PROOF. Let T be the tree constructed from G as in Fact 3.1. T can be constructed in linear time. Root T at an arbitrary leaf r. Note that r must be a block of G. To obtain a 7-coloring of G, we process the vertices of T that are 1-cut sets of G, in post-order. Consider the processing of a vertex c of T that is a 1-cut set of G. Let $b_1, \dots, b_k$ be the children of c in T, and f be the parent of c in T. Suppose that for each $1 \le i \le k$, we have obtained a 7-coloring $\mathcal{C}_i$ of the subgraph of G induced by the set of all vertices v such that v is contained in block b_i or a descendant of b_i in T. For each $1 \le i \le k$, let $V_{i,1}, \dots, V_{i,7}$ be the color classes of $\mathcal{C}_i$. Let $V_{0,1}, \dots, V_{0,7}$ be the color classes of the given 7-coloring of block f. By re-ordering if necessary, we can assume that $c \subseteq V_{i,1}$ for all $0 \le i \le k$. Now, the sets $\cup_{i=0}^{k} V_{i,1}, \dots, \cup_{i=0}^{k} V_{i,7}$ form a 7-coloring of the subgraph of G induced by the set of all vertices v such that v is contained in block f or a descendant of f in T. Thus, after the unique child of the root r is processed, we will obtain a 7-coloring of G. $\square$

Since the blocks of a graph can be computed in linear time [17], Corollary 3.2 implies a linear-time reduction from the problem of 7-coloring G to the problems of 7-coloring the blocks of G.

Next, suppose that G is 2-connected but not 3-connected. Let U be a 2-cut set of G, and V_1, ..., V_p be the vertex sets of the connected components of $G - U$. For $1 \le i \le p$, let G_i be the graph $G[V_i \cup U]$ if the two vertices in U are adjacent in G, while let G_i be the graph obtained from $G[V_i \cup U]$ by adding an edge between the two vertices in U otherwise. The graphs G_1, G_2, ..., G_p are called the *augmented components* of G induced by U. Obviously, all the augmented components of G induced by U are also 2-connected.

Lemma 3.3 *Suppose that G is 1-planar. Then, each augmented component of G induced by a 2-cut set U is also 1-planar. Moreover, for each augmented component G_i of G, we can obtain an edge-crossing list of G_i from L by removing all pairs (e, e') such that e or e' is not an edge of G_i.*

PROOF. Let $U = (u, v)$. The lemma is clear if u and v are adjacent in G. So, assume that they are not adjacent in G. Consider a 1-plane embedding $\mathcal{E}$ of G in which exactly the pairs of edges in L cross. Let G_i be an augmented component of G induced by U. G must have a path P between u and v such that no vertex of P other than u and v is in G_i.

We claim that no edge of P is crossed by an edge of G_i in $\mathcal{E}$. Towards a contradiction, assume that some edge $e = (x, y)$ of P is crossed by an edge $e' = (x', y')$ of G_i in $\mathcal{E}$. Let z be a vertex in $\{x, y\} - \{u, v\}$. Similarly, let z' be a vertex in $\{x', y'\} - \{u, v\}$. Since (u, v) is not an edge of G, z and z' exist. Note that no augmented component of G contains both z and z'. So, (z, z') can not be an edge of G. However, since $(e, e') \in L$, $\{x, y, x', y'\}$ induces a clique of G and (z, z') has to be an edge of G, a contradiction. Thus, the claim holds.

Now, suppose that we modify $\mathcal{E}$ as follows.

1. Delete all vertices and edges that are not contained in G_i or P.
2. Replace P by a single new edge between u and v. (Comment: By the above claim, the new edge between u and v is crossed by no edge of G_i.)

The modification yields a 1-plane embedding of G_i. Thus, the first assertion in the lemma holds. The second assertion is clear from the proof. $\square$

Replacing G by the augmented components induced by a 2-cut set is called *splitting* G. Suppose G is split, the augmented components are split, and so on, until no more splits are possible. The graphs constructed in this way are 3-connected and the set of the graphs are called a *2-decomposition* of G. (See Figure 3 for an example.) Each element of a 2-decomposition of G is called a *split component* of G. It is possible for G to have two or more 2-decompositions. A split component of G must be either a triangle or a 3-connected graph with at least four vertices.

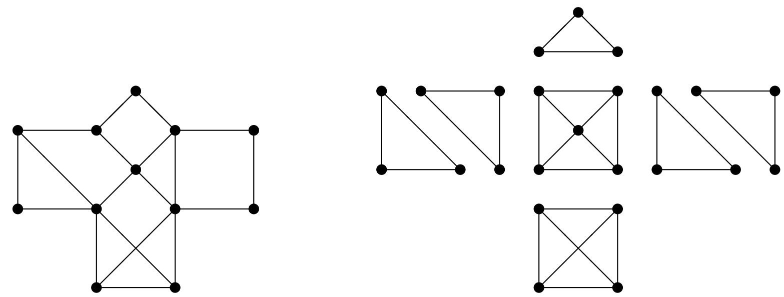


Figure 3: (1) A graph G (2) A 2-decomposition D of G

The following fact is widely known [18].

Fact 3.4 *Suppose that G is 2-connected but not 3-connected. Let D be a 2-decomposition of G. Let C be the family of all 2-cut sets of G used to split G into the split components in D. Consider the bipartite graph $T = (D \cup C, E_T)$ where E_T consists of all edges $\{d, c\}$ such that $d \in D$, $c \in C$, and the two vertices in c are vertices in d. Then, T is a tree.*

Corollary 3.5 *Suppose that G is 1-planar and connected but not 2-connected. Then, given a 2-decomposition D and a 7-coloring of each split component in D, we can compute a 7-coloring of G in linear time.*

Proof. The proof is similar to that of Corollary 3.5. Let T be the tree constructed from G as in Fact 3.4. T can be constructed in linear time. Root T at an arbitrary leaf r. Note that r must be a split component in D. To obtain a 7-coloring of G, we process the vertices of T that are 2-cut sets in C, in post-order. Consider the processing of a vertex $c \in C$ of T. Let $d_1, \ldots, d_k$ be the children of c in T, and f be the parent of c in T. Suppose that for each $1 \le i \le k$, we have obtained a 7-coloring $\mathcal{C}_i$ of the subgraph of G induced by the set of all vertices v such that v is contained in split component d_i or a descendant of d_i in T. For each $1 \le i \le k$, let $V_{i,1}, \ldots, V_{i,7}$ be the color classes of $\mathcal{C}_i$. Let $V_{0,1}, \ldots, V_{0,7}$ be the color classes of the given 7-coloring of split component f. The crucial point is that for each $0 \le i \le k$, the two vertices in c belong to different color classes among $V_{i,1}, \ldots, V_{i,7}$ because they are adjacent in each of the split components $f, d_1, \ldots, d_k$. So, by re-ordering if necessary, we can assume that one vertex in c belongs to $V_{i,1}$ for all $0 \le i \le k$, and the other belongs to $V_{i,2}$ for all $0 \le i \le k$. Now, the sets $\cup_{i=0}^{k} V_{i,1}, \ldots, \cup_{i=0}^{k} V_{i,7}$ form a 7-coloring of the subgraph of G induced by the set of all vertices v such that v is contained in split component f or a descendant of f in T. Thus, after the unique child of the root r is processed, we will obtain a 7-coloring of G. □

Since a 2-decomposition of a graph can be computed in linear time [13], Lemma 3.3 and Corollary 3.5 together imply a linear-time reduction from the problem of 7-coloring G to the problems of 7-coloring the split components in a 2-decomposition of G.

4 The Algorithm for the 3-Connected Case

Throughout this section, G denotes the input 3-connected 1-planar graph and L denotes the input edge-crossing list of G. By our discussion in Section 3, we may assume that for each pair $(e, e') \in L$, (1) e and e' share no endpoint, (2) the endpoints of e and e' induce a clique C of size 4 in G, and (3) no edge of C other than e and e' is contained in a pair in L.

Given G and L, we first construct a graph H as follows. H contains all vertices of G and all those edges of G that are contained in no pair in L. Moreover, for each (unordered) pair $\{e, e'\} \in L$, H contains a new vertex $v_{e,e'}$, and contains an edge between $v_{e,e'}$ and each endpoint of e and/or

e'. H does not contain other vertices or edges. Since L is an edge-crossing list of G, H is a planar graph. We then compute a plane embedding of H in linear time. For convenience, we identify H with its plane embedding. Hereafter, for a vertex v of H, we say that two neighbors u and w of v in H are *consecutive* if u and w appear around $v_{e,e'}$ consecutively (clockwise or counterclockwise) in H.

Fact 4.1 *Let $v_{e,e'}$ be a vertex in H but not in G. Suppose that x and y are two consecutive neighbors of $v_{e,e'}$ in H (note that $\{x, y\}$ is an edge in H by our assumptions on L). Then, the cycle C formed by the three edges $\{v_{e,e'}, x\}$, $\{x, y\}$, and $\{y, v_{e,e'}\}$ together is the boundary of some face of H.*

PROOF. For a contradiction, assume that the fact does not hold. Let U (respectively, W) be the set of all vertices of G that appear in the interior (respectively, exterior) of C in H. By the assumption, both U and W are nonempty. Moreover, $U \cup W \cup \{x, y\}$ is the vertex set of G. By planarity, it is easy to see that G has no edge $\{u, w\}$ with $u \in U$ and $w \in W$. Thus, $G - \{x, y\}$ is disconnected, contradicting the 3-connectivity of G. $\square$

Lemma 4.2 *We can modify H in linear time so that H satisfies the following conditions:*

1. *H has the same vertices as G, and contains all edges of G.*
2. *For each vertex v of H and for every two consecutive neighbors u and w of v in H, $\{u, w\}$ is an edge in H and crosses no edge in H.*
3. *For each pair of edges of H that cross in H, the endpoints of the two edges induce a clique of size 4 in H.*

PROOF. We modify H in linear time as follows:

1. For each face f of H whose boundary contains at least four vertices (all vertices on the boundary of f must be vertices of G by Fact 4.1), triangulate f by adding enough edges to H in such a way that H remains to be a simple graph. [*Comment*: Since G is 3-connected and H is planar during this step, no two vertices on the boundary of f that do not appear consecutively on the boundary can be adjacent in H. So, it is easy to triangulate f. Moreover, after this step, every face of H is a triangle and hence Condition 2 is satisfied.]

2. For each vertex $v_{e,e'}$ in H but not in G, delete $v_{e,e'}$ and the four edges adjacent to it from H, and further add the two edges e and e' of G to H in such a way that e and e' cross in the plane at the point where $v_{e,e'}$ was located. (*Comment:* By planarity, it is easy to see that at most one of e and e' was already in H before this step. Also note that for every two vertices v and u in H, at most one edge between v and u can be added during this step, because the pairs in L are pairwise disjoint. Thus, after this step, for every two vertices v and u in H, there are at most two edges between v and u in H. Moreover, after this step, Condition 2 remains satisfied (as can be seen from Figure 4(1)), and Condition 3 becomes satisfied.]

3. For each vertex v in H and for each neighbor u of v in H, if there are two edges e_1 and e_2 between v and u in H, then delete the edge $e_i \in \{e_1, e_2\}$ from H such that e_i was added to H in Step 2. [*Comment:* The time needed for v in this step is $O(d_H(v))$. Moreover, by the first sentence in the comment on Step 2, Condition 2 remains satisfied after this step (as can be seen from Figure 4(2)).]

After the above steps, H satisfies the conditions in the lemma. $\square$

Corollary 4.3 *Every 1-planar graph has a supergraph that is a 4-map graph.*

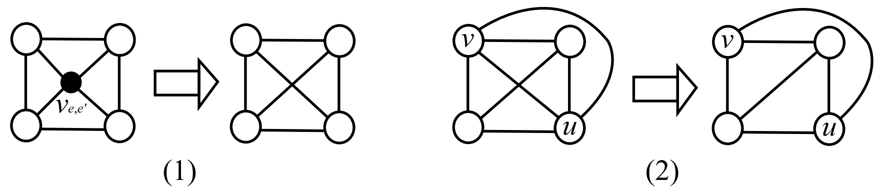


Figure 4: (1) Removing a vertex $v_{e,e'}$. (2) Removing a duplicated edge.

Proof. By the definition of a 4-map graph, it is easy to see that a graph K is a 4-map graph if it has a 1-plane embedding $\mathcal{E}$ such that whenever two edges of K cross in $\mathcal{E}$, their endpoints induce a clique of size 4 in K. Now, the corollary follows from Lemma 4.2 immediately. □

Hereafter, without loss of generality (by Lemma 4.2), we make the following assumption:

Assumption 4.4 *H satisfies the conditions in Lemma 4.2.*

Now, by Condition 1 in Lemma 4.2, it suffices to 7-color H in order to 7-color G. Hereafter, we will work on H instead of G.

Corollary 4.5 *The following two statements hold:*

1. *Let v be a vertex in H. Suppose that u is a neighbor of v in H such that edge $\{v, u\}$ crosses another edge $\{x, y\}$ in H. Then, $\{x, y\} \subseteq N_H(v)$, and x, u, y appear around v consecutively in H in this order (clockwise or counterclockwise).*

2. *Let u and w be two consecutive neighbors of v in H. Then, at least one of edges $\{v, u\}$ and $\{v, w\}$ crosses no edge in H.*

Proof. We first prove Statement 1. By Condition 3 in Lemma 4.2, $\{x, y\} \subseteq N_H(v)$. For a contradiction, assume that x and u are not consecutive neighbors of v in H. Then, there is a $w \in N_H(v)$ such that u and w are consecutive neighbors of v in H and u, w, x appear around v in H in this order clockwise or counterclockwise (see Figure 5(1)). Note that w and x may be not consecutive neighbors of v in H. By Condition 2 in Lemma 4.2, $\{u, w\}$ is an edge in H and it crosses no edge in H. However, this is impossible (as can be seen from Figure 5(1)).

We next prove Statement 2. Suppose that $\{v, u\}$ crosses an edge e in H. Then, by Statement 1, $e = \{w, x\}$ where x is the vertex in $N_H(v) - \{w\}$ such that u and x are consecutive neighbors of v in H. Figure 5(2) illustrates the sub-embedding of H induced by $\{v, u, w, x\}$. Note that $\{w, u\}$ and $\{u, x\}$ are edges in H by Condition 2 in Lemma 4.2. If edge $\{v, w\}$ also crossed an edge e' in H, then by Figure 5(2) and Condition 3 in Lemma 4.2, there would be a vertex $y \in N_H(v) - \{u, w, x\}$ such that w, y, u appear around v consecutively in H in this order clockwise, contradicting that u and w are consecutive neighbors of v in H. □

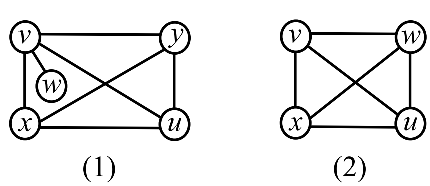


Figure 5: (1) Vertex v and some of its neighbors in H. (2) The sub-embedding of H induced by $\{v, u, w, x\}$.

4.1 A Structure Lemma

Fix two constants α and K with $1 < \alpha < 2$ and $K > 7 + 9/(\alpha - 1)$. Let v be a vertex of H. If $|d_H(v)| \le K$, we say that v is *small*; otherwise, we say that v is *large*. We say that v is *reducible* if one of the following holds:

1. $d_H(v) \le 6$.
2. $d_H(v) = 7$ and $N_H(v)$ contains at most one large vertex.
3. $d_H(v) = 8$, $N_H(v)$ contains no large vertex, and one of the following holds:
   (a) There are at most two vertices $u \in N_H(v)$ with $d_H(u) \ge 9$.
   (b) There are exactly three vertices $u \in N_H(v)$ with $d_H(u) \ge 9$ and there are distinct vertices u_1, u_2, u_3 in $N_H(v)$ such that $d_H(u_1) \ge 9$, $d_H(u_2) \ge 9$, $d_H(u_3) \le 8$, and $\{v, u_2\}$ and $\{u_1, u_3\}$ are edges of H and they cross in H. (See Figure 6.)

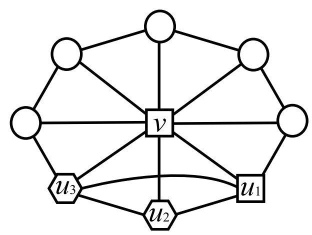


Figure 6: A reducible vertex v and its neighbors, when $d_H(v) = 8$ and v has exactly three vertices of degree ≥ 9. In this figure, $d_H(u_2) \ge 9$, $d_H(u_3) \ge 9$, and $d_H(u_1) \le 8$.

Lemma 4.6 *Let R be the set of reducible vertices in H. Then, R contains a constant fraction of vertices of H.*

PROOF. Since H is 3-connected, each vertex of H is of degree at least 3. For each $j \ge 3$, let n_j be the number of vertices v in H with $d_H(v) = j$. Let $\hat{n}_7$ (respectively, $\hat{n}_8$) be the number of reducible vertices v of H with $d_H(v) = 7$ (respectively, $d_H(v) = 8$). Let $\bar{n}_8$ be the number of vertices v of H such that $d_H(v) = 8$ and $N_H(v)$ contains no large vertex.

Let n be the number of vertices in H. Since H is a 4-map graph, it has at most $4n - 8$ edges [6]. Thus, $\sum_{j=3}^{n-1} jn_j \le 8n - 16$. Isolating the terms containing n_7, we have $n_7 > \sum_{j=3}^{6}(j-8)n_j + \sum_{j=8}^{n-1}(j-8)n_j$. In turn,

$$n_7 > \sum_{j=8}^{n-1}(j-8)n_j - 5\sum_{j=3}^{6} n_j. \tag{1}$$

Let m' be the number of edges $\{u, v\}$ in H such that at least one of u and v is large. By the definitions of $\hat{n}_7$ and $\bar{n}_8$, $m' \ge 2(n_7 - \hat{n}_7) + (n_8 - \bar{n}_8)$. On the other hand, $m' \le \sum_{j=K+1}^{n-1} jn_j$. Therefore, $2(n_7 - \hat{n}_7) + (n_8 - \bar{n}_8) \le \sum_{j=K+1}^{n-1} jn_j$, or equivalently,

$$2\hat{n}_7 + \bar{n}_8 \ge 2n_7 + n_8 - \sum_{j=K+1}^{n-1} jn_j. \tag{2}$$

Now, adding α times Inequality (1) to Inequality (2), we have

$$2\hat{n}_7 + \bar{n}_8 > (2-\alpha)n_7 + n_8 - \sum_{j=K+1}^{n-1} jn_j + \alpha\sum_{j=8}^{n-1}(j-8)n_j - 5\alpha\sum_{j=3}^{6} n_j.$$

In turn, since $\alpha > 1$, we have

$$5\alpha \sum_{j=3}^{6} n_j + 2\hat{n}_7 + \bar{n}_8 > (2-\alpha)n_7 + n_8 + \alpha \sum_{j=9}^{K} n_j + \sum_{j=K+1}^{n-1} ((\alpha-1)(K+1) - 8\alpha)n_j. \tag{3}$$

Let X be the set of all vertices v in H such that $d_H(v) = 8$, $N_H(v)$ contains no large vertex, and $N_H(v)$ contains at least four vertices u with $d_H(u) \geq 9$. Let Y be the set of all vertices v in H such that $d_H(v) = 8$, $N_H(v)$ contains no large vertex, $N_H(v)$ contains exactly three vertices u with $d_H(u) \geq 9$, and $N_H(v)$ does not contain distinct vertices u_1, u_2, u_3 such that $d_H(u_1) \geq 9$, $d_H(u_2) \geq 9$, $d_H(u_3) \leq 8$, and $\{v, u_2\}$ and $\{u_1, u_3\}$ are edges of H and they cross in H. Then,

$$|X| + |Y| = \bar{n}_8 - \hat{n}_8. \tag{4}$$

Consider the bipartite graph $B = (X \cup Y \cup Z, E_B)$, where Z is the set of all vertices u in H such that $d_H(u) \geq 9$ and $N_H(u) \cap (X \cup Y) \neq \emptyset$, and E_B consists of all edges $\{v, u\}$ in H such that $v \in X \cup Y$ and $u \in Z$. We want to bound $|E_B|$ from above. To this end, first note that B inherits a 1-plane embedding $\mathcal{B}$ from H. Let P be the set of all unordered pairs of edges of B that cross in $\mathcal{B}$. A crucial point is that if we modify $\mathcal{B}$ by deleting exactly one edge in each pair in P, then we get a bipartite planar graph. By this and the well-known fact that a bipartite planar graph with $|X| + |Y| + |Z|$ vertices can have at most $2(|X| + |Y| + |Z|) - 4$ edges, we have

$$|E_B| \leq 2(|X| + |Y| + |Z|) - 4 + |P|. \tag{5}$$

We next bound $|P|$ from above. Consider the graph $\mathcal{B}_P$ constructed from $\mathcal{B}$ as follows. The endpoints of the edges in the pairs in P are vertices of $\mathcal{B}_P$. Moreover, for each (unordered) pair $\{e, e'\} \in P$, $\mathcal{B}_P$ contains a new vertex $v_{e,e'}$, and contains an edge between $v_{e,e'}$ and each endpoint of e and/or e'. $\mathcal{B}_P$ does not contain other vertices or edges. Obviously, $\mathcal{B}_P$ is a bipartite planar graph and has exactly $4|P|$ edges. On the other hand, no vertex in Y can be a vertex of $\mathcal{B}_P$ by the definition of Y and Condition 3 in Lemma 4.2; hence $\mathcal{B}_P$ has at most $|X| + |Z| + |P|$ vertices, and in turn has at most $2(|X| + |Z| + |P|) - 4$ edges. Therefore,

$$4|P| \leq 2(|X| + |Z| + |P|) - 4. \tag{6}$$

Combining Inequalities (5) and (6), we get

$$|E_B| \leq 3(|X| + |Z|) + 2|Y| - 6. \tag{7}$$

On the other hand, $|E_B| \geq 4|X| + 3|Y|$ by the construction of graph B. By this and Inequality (7), we have

$$|X| + |Y| \leq 3|Z| - 6. \tag{8}$$

By the definition of Z, $|Z| \leq \sum_{j=9}^{n-1} n_j$. So, Inequalities (4) and (8) imply $\bar{n}_8 - \hat{n}_8 \leq 3\sum_{j=9}^{n-1} n_j - 6$, or equivalently

$$\hat{n}_8 \geq \bar{n}_8 - 3\sum_{j=9}^{n-1} n_j + 6. \tag{9}$$

Now, adding 1/3 times Inequality (9) to Inequality (3) and further rearranging, we have

$$\begin{aligned}
& 5\alpha \sum_{j=3}^{6} n_j + 2\hat{n}_7 + \frac{1}{3}\hat{n}_8 \\
> \quad & (2-\alpha)n_7 + (n_8 - \frac{2}{3}\bar{n}_8) + (\alpha - 1)\sum_{j=9}^{K} n_j + \sum_{j=K+1}^{n-1} ((\alpha-1)(K+1) - 8\alpha - 1)n_j \\
\geq \quad & (2-\alpha)n_7 + \frac{1}{3}n_8 + (\alpha - 1)\sum_{j=9}^{K} n_j + \sum_{j=K+1}^{n-1} ((\alpha-1)(K+1) - 8\alpha - 1)n_j.
\end{aligned}$$

Now, we choose $\alpha = 5/3$ and $K = 21$. Then,

$$\frac{25}{3}\sum_{j=3}^{6} n_j + 2\hat{n}_7 + \frac{1}{3}\hat{n}_8 > \frac{1}{3}n_7 + \frac{1}{3}n_8 + \frac{2}{3}\sum_{j=9}^{K} n_j + \frac{1}{3}\sum_{j=K+1}^{n-1} n_j.$$

Consequently,

$$\sum_{j=3}^{6} n_j + \hat{n}_7 + \hat{n}_8 > \frac{1}{26}\sum_{j=3}^{n-1} n_j.$$

This completes the proof of the lemma. □

Corollary 4.7 *We can compute a set I of reducible vertices of H in linear time such that the following conditions are satisfied:*

1. *I contains a constant fraction of vertices of H.*
2. *For every two vertices u and v in I, there is no path P between u and v in H such that P has at most three edges and has no large vertex.*

Proof. Consider a graph $H_R = (R, E_R)$ as follows. R is as in Lemma 4.6. For every two vertices u and v in R such that there is a path P between u and v in H such that P has at most three edges and has no large vertex, there is an edge between u and v in H_R. H_R contains no other edges. We set I to be a maximal independent set of H_R. Obviously, H_R and I can be computed in linear time.

Consider a vertex $v \in R$. If $d_H(v) \le 6$, then $d_{H_R}(v) \le 6 \times 20 \times 20 = 2400$. If $d_H(v) = 7$, then $d_{H_R}(v) \le 7 \times 20 \times 20 = 2800$. If $d_H(v) = 8$, then $d_{H_R}(v) \le 8 \times 20 \times 20 = 3200$. So, H_R is a graph with maximum degree 3200. In turn, since I is a maximal independent set of H_R, $|I| \ge |R|/3201$. Therefore, by Lemma 4.6, I contains a constant fraction of vertices in H. □

4.2 Outline of the Algorithm

We first give an outline of the algorithm. It first computes a set I of reducible vertices of H satisfying the conditions in Corollary 4.7. It then uses I and H to construct a new 1-planar graph G' in linear time such that the number of vertices in G' is a constant fraction of the number of vertices in H and a 7-coloring of H can be constructed in linear time from an arbitrarily given 7-coloring of G'. It further recurses on G' to obtain a 7-coloring of G' which is then used to obtain a 7-coloring of H in linear time. Since each recursion takes linear time and reduces the size of the graph by a constant fraction, the overall time is linear. The core of the algorithm is in the construction of G'.

4.3 Constructing Graph G' for Recursion

To construct G', we may simply remove all $v \in I$ with $d_H(v) \le 6$ from H because each 7-coloring of $H - v$ extends to a 7-coloring of H. Similarly, for each $v \in I$ such that $N_H(v)$ contains a vertex u with $d_H(u) \le 6$, we may remove u from H. However, these are not enough because I may contain very few such vertices v. So, we need to do something about those vertices $v \in I$ such that $7 \le d_H(v) \le 8$ and $N_H(v)$ contains no vertex u with $d_H(u) \le 6$. We call such vertices v *critical* vertices. The idea is to explore the neighborhood structure of critical vertices. First, we need the following definitions:

Definition 4.1 *A vertex x in H is* dangerous *for a critical vertex v if one of the following holds:*

- *$d_H(v) = 7$ and x is a large neighbor of v in H.*

- *$d_H(v) = 8$, $x \notin N_H(v) \cup \{v\}$, and x is adjacent to some vertex $u \in N_H(v)$ in H.*

Note that a vertex x may be dangerous for more than one critical vertex.

Definition 4.2 *Let v be a critical vertex with $d_H(v) = 7$. A* mergable pair *for v is a pair (u, w) of two nonadjacent neighbors of v in H such that u is small and the graph G_1 obtained from $H - v$ by merging $\langle u, w\rangle$ is a 1-planar graph.*

In Definition 4.2, w may be dangerous for v. Moreover, no matter whether w is dangerous for v, w remains in G_1. The intuition behind Definition 4.2 is that we can extend a given 7-coloring of G_1 to a 7-coloring of H as follows: Let u have the color of w, and then let v have a color that is assigned to no vertex in $N_H(v)$.

Definition 4.3 *Let v be a critical vertex with $d_H(v) = 8$.*

1. *A* mergable triple *for v is a set $\{u_1, u_2, u_3\}$ of three pairwise nonadjacent neighbors of v in H such that the graph G_2 obtained from $H - v$ by merging $\langle u_1, u_2, u_3\rangle$ is a 1-planar graph.*

2. *Two* simultaneously mergable pairs *for v are two pairs (u_1, u_2) and (w_1, w_2) such that u_1, u_2, w_1, and w_2 are distinct neighbors of v in H, neither $\{u_1, u_2\}$ nor $\{w_1, w_2\}$ is an edge of H, and the graph G_3 obtained from $H - v$ by merging $\langle u_1, u_2\rangle$ and merging $\langle w_1, w_2\rangle$ is a 1-planar graph.*

3. *A* desired quadruple *for v is an ordered list (u, w_1, w_2, w_3) of four distinct neighbors in H such that*
   - *$d_H(u) \le 8$,*
   - *$\{w_1, w_2\} \subseteq N_H(u)$, and*
   - *$\{w_1, w_2, w_3\}$ is an independent set in H, and the graph G_4 obtained from $H - \{v, u\}$ by merging $\langle w_1, w_2, w_3\rangle$ is a 1-planar graph.*

4. *A* favorite quintuple *for v is an ordered list (u, w_1, w_2, w_3, w_4) of five distinct neighbors of v in H such that*
   - *$d_H(u) \le 8$,*
   - *$\{w_1, w_2\} \subseteq N_H(u)$, and*
   - *neither $\{w_1, w_2\}$ nor $\{w_3, w_4\}$ is an edge in H, and the graph G_5 obtained from $H - \{v, u\}$ by merging $\langle w_1, w_2\rangle$ and merging $\langle w_3, w_4\rangle$ is a 1-planar graph.*

5. *A* desired quintuple *for v is an ordered list $(u_1, u_2, w_1, w_2, w_3)$ of five distinct neighbors of v in H such that*
   - *$d_H(u_1) \le 8$ and $d_H(u_2) \le 8$,*
   - *$\{w_1, w_2\} \subseteq N_H(u_1)$ and $\{w_2, w_3\} \subseteq N_H(u_2)$, and*
   - *$\{w_1, w_2, w_3\}$ is an independent set in H, and the graph G_6 obtained from $H - \{v, u_1, u_2\}$ by merging $\langle w_1, w_2, w_3\rangle$ is a 1-planar graph.*

6. *A* desired sextuple *for v is an ordered list $(u_1, u_2, w_1, w_2, w_3, w_4)$ of six distinct neighbors of v in H such that*
   - *$d_H(u_1) \le 8$ and $d_H(u_2) \le 8$,*
   - *$\{w_1, w_2\} \subseteq N_H(u_1)$ and $\{w_3, w_4\} \subseteq N_H(u_2)$, and*
   - *neither $\{w_1, w_2\}$ nor $\{w_3, w_4\}$ is an edge in H, and the graph G_7 obtained from $H - \{v, u_1, u_2\}$ by merging $\langle w_1, w_2\rangle$ and merging $\langle w_3, w_4\rangle$ is a 1-planar graph.*

7. *A* useful sextuple *for v is an ordered list $(u_1, u_2, w_1, x_1, w_2, x_2)$ of six distinct vertices in H such that*

   - *$d_H(u_1) \le 8$, $d_H(u_2) \le 8$, and $\{u_1, u_2\} \subseteq \{v\} \cup N_H(v)$,*
   - *$w_1 \in \{v\} \cup N_H(v)$ and $w_2 \in \{v\} \cup N_H(v)$,*
   - *$\{w_1, x_1, w_2, x_2\} \subseteq N_H(u_1)$ and $\{u_1, w_1, x_1\} \subseteq N_H(u_2)$, and*
   - *neither $\{w_1, x_1\}$ nor $\{w_2, x_2\}$ is an edge in H, and the graph G_8 obtained from $H - \{u_1, u_2\}$ by merging $\langle w_1, x_1\rangle$ and merging $\langle w_2, x_2\rangle$ is a 1-planar graph.*

8. *A* useful triple *for v is an ordered list (u, w, x) of three distinct vertices in H such that*

   - *$d_H(u) \le 7$ and $u \in N_H(v)$,*
   - *$w \in N_H(v)$ and $\{w, x\} \subseteq N_H(u)$, and*
   - *$\{w, x\}$ is not an edge in H and the graph G_9 obtained from $H - \{u\}$ by merging $\langle w, x\rangle$ is a 1-planar graph.*

In Definition 4.3(7), x_1 and x_2 may be dangerous for v. Also, in Definition 4.3(8), x may be dangerous for v.

The intuitions behind Definitions 4.3(1) and 4.3(2) are similar to that of Definition 4.2. The intuition behind Definition 4.3(3) is that we can extend a given 7-coloring of the graph G_4 to a 7-coloring of H as follows: let w_1 and w_2 have the color of w_3, let u have a color assigned to no vertex in $N_H(u) - \{v\}$, and further let v have a color assigned to no vertex in $N_H(v)$ before.

The intuition behind Definition 4.3(4) is that we can extend a given 7-coloring of the graph G_5 to a 7-coloring of H as follows: let w_1 have the color of w_2, let w_3 have the color of w_4, let u have a color assigned to no vertex in $N_H(u) - \{v\}$, and further let v have a color assigned to no vertex in $N_H(v)$ before.

The intuition behind Definition 4.3(5) is that we can extend a given 7-coloring of the graph G_6 to a 7-coloring of H as follows: let w_1 and w_2 have the color of w_3, let u_2 have a color assigned to no vertex in $N_H(u_2) - \{v\}$, let u_1 have a color assigned to no vertex in $N_H(u_1) - \{v\}$, and further let v have a color assigned to no vertex in $N_H(v)$ before.

The intuition behind Definition 4.3(6) is that we can extend a given 7-coloring of the graph G_7 to a 7-coloring of H as follows: let w_1 have the color of w_2, let w_3 have the color of w_4, let u_2 have a color assigned to no vertex in $N_H(u_2) - \{v\}$, let u_1 have a color assigned to no vertex in $N_H(u_1) - \{v\}$, and further let v have a color assigned to no vertex in $N_H(v)$ before.

The intuition behind Definition 4.3(7) is that we can extend a given 7-coloring of the graph G_8 to a 7-coloring of H as follows: let w_1 have the color of x_1, let w_2 have the color of x_2, let u_2 have a color assigned to no vertex in $N_H(u_2) - \{u_1\}$ before, and further let u_1 have a color assigned to no vertex in $N_H(u_1)$ before.

Finally, the intuition behind Definition 4.3(8) is that we can extend a given 7-coloring of the graph G_9 to a 7-coloring of H as follows: let w have the color of x, and then let u have a color assigned to no vertex in $N_H(u)$ before.

The following theorem can be proved by a case-analysis. Since the proof is very tedious, we postpone the proof to Section 5.

Theorem 4.8 *For a critical vertex v, call an edge e in H a* basic critical edge *for v if at least one endpoint of e is v or a small neighbor of v in H, Moreover, for a critical vertex v, call an edge e in H a* critical edge *for v if e is a basic critical edge for v or e crosses a basic critical edge for v in H. Then, for every critical vertex v, the following hold:*

1. *If $d_H(v) = 7$, then we can use the sub-embedding of H induced by the set of critical edges for v to find a mergable pair for v in $O(1)$ time such that the graph G_1 defined in Definition 4.2 has a 1-plane embedding H' satisfying the following three conditions:*

(C1) *For every pair of edges* e_1 *and* e_2 *in* H', e_1 *and* e_2 *cross each other in embedding* H' *if and only if they cross each other in embedding* H.

(C2) *For every vertex* x *in* H' *that is neither* v *nor a small neighbor of* v *in* H, *and for every sequence* $\langle e_1, \ldots, e_k \rangle$ *of edges in* H *that are incident to* x *but incident to neither* v *nor a small neighbor of* v, *if edges* $e_1, \ldots, e_k$ *appear around* x *consecutively in this order in embedding* H, *then edges* $e_1, \ldots, e_k$ *appear around* x *consecutively in this order in embedding* H'.

(C3) *Same as (C2) but with both occurrences of the word "consecutively" deleted.*

2. *If* $d_H(v) = 8$, *then we can use the sub-embedding of* H *induced by the set of critical edges for* v *to compute one of the following for* v *in* $O(1)$ *time:*
   - *A mergable triple such that the graph* G_2 *defined in Definition 4.3(1) has a 1-plane embedding* H' *satisfying the above conditions (C1) through (C3).*
   - *Two simultaneously mergable pairs such that the graph* G_3 *defined in Definition 4.3(2) has a 1-plane embedding* H' *satisfying the above conditions (C1) through (C3).*
   - *A desired quadruple such that the graph* G_4 *defined in Definition 4.3(3) has a 1-plane embedding* H' *satisfying the above conditions (C1) through (C3).*
   - *A favorite quintuple such that the graph* G_5 *defined in Definition 4.3(4) has a 1-plane embedding* H' *satisfying the above conditions (C1) through (C3).*
   - *A desired quintuple such that the graph* G_6 *defined in Definition 4.3(5) has a 1-plane embedding* H' *satisfying the above conditions (C1) through (C3).*
   - *A desired sextuple such that the graph* G_7 *defined in Definition 4.3(6) has a 1-plane embedding* H' *satisfying the above conditions (C1) through (C3).*
   - *A useful sextuple such that the graph* G_8 *defined in Definition 4.3(7) has a 1-plane embedding* H' *satisfying the above conditions (C1) through (C3).*
   - *A useful triple such that the graph* G_9 *defined in Definition 4.3(8) has a 1-plane embedding* H' *satisfying the above conditions (C1) through (C3).*

In the constructions of graphs G_1 through G_9 (cf. Definitions 4.2 and 4.3), we may merge a dangerous vertex for v only with a small vertex in $\{v\} \cup N_H(v)$ (and the dangerous vertex remains after the merging operation), may delete only v and/or some small vertices in $N_H(v)$, and may touch only some critical edges for v. Note that the set of critical edges for a critical vertex is disjoint from the set of critical edges for another critical vertex, because of the second condition in Corollary 4.7. Thus, Conditions (C1) through (C3) together guarantee that for each critical vertex v, we can find and use a mergable pair, a mergable triple, two simultaneously mergable pairs, a desired quadruple, a favorite quintuple, a desired quintuple, a desired sextuple, a useful sextuple, or a useful triple for v to modify H in such a way that after the modification, we can still find a mergable pair, a mergable triple, two simultaneously mergable pairs, a desired quadruple, a favorite quintuple, a desired quintuple, a desired sextuple, a useful sextuple, or a useful triple for each other critical vertex.

Now, we are ready to explain how to construct G'. The construction of G' from H is done as follows.

1. For each critical vertex v with $d_H(v) = 7$, find a mergable pair for v as guaranteed in Theorem 4.8.

2. For each critical vertex v with $d_H(v) = 8$, find a mergable triple, two simultaneously mergable pairs, a desired quadruple, a favorite quintuple, a desired quintuple, a desired sextuple, a useful sextuple, or a useful triple for v as guaranteed in Theorem 4.8.

3. For each critical vertex v with $d_H(v) = 7$ and the mergable pair (u, w) found for v in Step 1, remove v from H and further merge $\langle u, w\rangle$.

4. For each critical vertex v with $d_H(v) = 8$, perform the following:

   (a) If a mergable triple $\{u_1, u_2, u_3\}$ was found for v in Step 2, then remove v from H and further merge $\langle u_1, u_2, u_3\rangle$.

   (b) If two simultaneously mergable pairs (u_1, u_2) and (w_1, w_2) were found for v in Step 2, then remove v from H, merge $\langle u_1, u_2\rangle$, and further merge $\langle w_1, w_2\rangle$.

   (c) If a desired quadruple (u, w_1, w_2, w_3) was found for v in Step 2, then remove v and u from H, and further merge $\langle w_1, w_2, w_3\rangle$.

   (d) If a favorite quintuple (u, w_1, w_2, w_3, w_4) was found for v in Step 2, then remove v and u from H, merge $\langle w_1, w_2\rangle$, and further merge $\langle w_3, w_4\rangle$.

   (e) If a desired quintuple $(u_1, u_2, w_1, w_2, w_3)$ was found for v in Step 2, then remove v, u_1, and u_2 from H, and further merge $\langle w_1, w_2, w_3\rangle$.

   (f) If a desired sextuple $(u_1, u_2, w_1, w_2, w_3, w_4)$ was found for v in Step 2, then remove v, u_1, and u_2 from H, merge $\langle w_1, w_2\rangle$, and further merge $\langle w_3, w_4\rangle$.

   (g) If a useful sextuple $(u_1, u_2, w_1, x_1, w_2, x_2)$ was found for v in Step 2, then remove u_1 and u_2 from H, merge $\langle w_1, x_1\rangle$, and further merge $\langle w_2, x_2\rangle$.

   (h) If a useful triple (u, w, x) was found for v in Step 2, then remove u from H and further merge $\langle w, x\rangle$.

5. Remove all $v \in I$ with $d_H(v) \le 6$ from H.

6. For each $v \in I$ such that $N_H(v)$ contains a vertex u with $d_H(u) \le 6$, remove all such vertices u from H.

By the discussion in the paragraph succeeding Theorem 4.8, the merging and removal operations in Steps 3 through 6 do not interfere with each other. It is also easy to see that the construction of G' takes $O(|I|)$ time (and hence linear time).

Recall that a vertex x may be dangerous for more than one critical vertex. So, during the construction of G', it is possible that after a dangerous vertex x for some critical vertex v is merged with a small vertex in $\{v\} \cup N_H(v)$, x is merged with a small vertex in $\{v'\} \cup N_H(v')$ later, where $v' \neq v$ is a critical vertex for which x is dangerous too. Fortunately, the second condition in Corollary 4.7 guarantees that during the construction of G', adjacent vertices are never merged together.

By Condition (C1), G' has a 1-plane embedding H'' such that for every pair of edges e_1 and e_2 in G', e_1 and e_2 cross each other in embedding H'' if and only if they cross each other in embedding H. Thus, we can compute a list L' of disjoint (unordered) pairs of edges of G' in linear time such that G' has a 1-plane embedding in which the two edges in each pair in L' cross while no two other edges of G' cross.

4.4 Detailed Discription of the Overall Algorithm

Let G be the given input graph and L be the given list of disjoint (unordered) pairs of edges of G such that G has a 1-plane embedding in which the two edges in each pair in L cross while no two other edges of G cross. The algorithm proceeds as follows.

1. If G is not connected, then recursively 7-color the vertices of each connected component of G, and combine the colorings into a 7-coloring of G in a straightforward way.

2. If G is not 2-connected, then recursively 7-color the vertices of each block of G, and combine the colorings into a 7-coloring of G as in Corollary 3.2.

3. If G is not 3-connected, then compute a 2-decomposition D, recursively 7-color each split component in D, and further combine the colorings into a 7-coloring of G as in Corollary 3.5.
4. If G is 3-connected, then perform the following:
   (a) Use G and L to construct a 1-plane embedding H as in Lemma 4.2.
   (b) Compute a set I of vertices in H as in Corollary 4.7.
   (c) Use H and I to construct a 1-planar graph G' and a list L' of disjoint (unordered) pairs of edges of G' such that G' has a 1-plane embedding in which the two edges in each pair in L' cross while no two other edges of G' cross. (See Section 4.3 for details.)
   (d) Recursively 7-color G'.
   (e) For each critical vertex $v \in I$ with $d_H(v) = 8$, perform the following:
      i. If a mergable triple $\{u_1, u_2, u_3\}$ was found for v in Step 4c, then (one of u_1, u_2, and u_3 remains in G'), let u_1, u_2, and u_3 have the same color assigned to one of them in Step 4d, and further let v have a color assigned to no vertex in $N_H(v)$ before.
      ii. If two simultaneously mergable pairs (u_1, u_2) and (w_1, w_2) were found for v in Step 4c, then (u_2 and w_2 remain in G'), let u_1 have the color of u_2, let w_1 have the color of w_2, and further let v have a color assigned to no vertex in $N_H(v)$ before.
      iii. If a desired quadruple (u, w_1, w_2, w_3) was found for v in Step 4c, then (w_3 remains in G'), let w_1 and w_2 have the color of w_3, let u have a color assigned to no vertex in $N_H(u) - \{v\}$ before, and further let v have a color assigned to no vertex in $N_H(v)$ before.
      iv. If a favorite quintuple (u, w_1, w_2, w_3, w_4) was found for v in Step 4c, then (w_2 and w_4 remain in G'), let w_1 have the color of w_2, let w_3 have the color of w_4, let u have a color assigned to no vertex in $N_H(u) - \{v\}$ before, and further let v have a color assigned to no vertex in $N_H(v)$ before.
      v. If a desired quintuple $(u_1, u_2, w_1, w_2, w_3)$ was found for v in Step 4c, then (w_3 remains in G'), let w_1 and w_2 have the color of w_3, let u_2 have a color assigned to no vertex in $N_H(u_2) - \{v\}$ before, let u_1 have a color assigned to no vertex in $N_H(u_1) - \{v\}$ before, and further let v have a color assigned to no vertex in $N_H(v)$ before.
      vi. If a desired sextuple $(u_1, u_2, w_1, w_2, w_3, w_4)$ was found for v in Step 4c, then (w_2 and w_4 remain in G'), let w_1 have the color of w_2, let w_3 have the color of w_4, let u_2 have a color assigned to no vertex in $N_H(u_2) - \{v\}$ before, let u_1 have a color assigned to no vertex in $N_H(u_1) - \{v\}$ before, and further let v have a color assigned to no vertex in $N_H(v)$ before.
      vii. If a useful sextuple $(u_1, u_2, w_1, x_1, w_2, x_2)$ was found for v in Step 4c, then (x_1 and x_2 remain in G'), let w_1 have the color of x_1, let w_2 have the color of x_2, let u_2 have a color assigned to no vertex in $N_H(u_2) - \{u_1\}$ before, and further let u_1 have a color assigned to no vertex in $N_H(u_1)$ before.
      viii. If a useful triple (u, w, x) was found for v in Step 4c, then (x remains in G'), let w have the color of x, and further let u have a color assigned to no vertex in $N_H(u)$ before.
   (f) For each critical vertex $v \in I$ with $d_H(v) = 7$ and for the mergable pair (u, w) found for v in Step 4c, let u have the color of w, and further let v have a color assigned to no vertex in $N_H(v)$ before.
   (g) For each $v \in I$ with $d_H(v) \leq 6$, let v have a color assigned to no vertex in $N_H(v)$ before.
   (h) For each $v \in I$ such that $N_H(v)$ contains a vertex u with $d_H(u) \leq 6$, color the vertices in $N_H(v) - N_{G'}(v)$ in an arbitrary order in such a way that each such vertex u gets a color assigned to no vertex in $N_H(u)$ before.
5. Output the 7-coloring of H obtained.

The correctness of the algorithm is clear. It is also obvious that the algorithm takes linear time.

5 Proof of Theorem 4.8

Throughout this section, fix a critical vertex $v \in I$, and let E_H be the set of edges in H. Let $v_1, \ldots, v_k$ be the vertices in $N_H(v)$, and assume that they appear around v in H in this order clockwise.

We define two functions $f_s : N_H(v) \to N_H(v)$ and $f_p : N_H(v) \to N_H(v)$ as follows. For each vertex $v_i \in N_H(v)$, $f_s(v_i) = v_{i+1}$ if $1 \le i \le k-1$, while $f_s(v_i) = v_1$ if $i = k$. Similarly, for each vertex $v_i \in N_H(v)$, $f_p(v_i) = v_{i-1}$ if $2 \le i \le k$, while $f_p(v_i) = v_k$ if $i = 1$. As usual, for each function f among f_s and f_p and for each integer $h \ge 1$, we define the function $f^h : N_H(v) \to N_H(v)$ inductively as follows. For each $v_i \in N_H(v)$, $f^h(v_i) = f(v_i)$ if $h = 1$, while $f^h(v_i) = f(f^{h-1}(v_i))$ otherwise.

By Condition 2 in Lemma 4.2, $\{\{v_i, f_s(v_i)\} \mid v_i \in N_H(v)\}$ is a subset of E_H and the edges in this subset altogether form a cycle C_v in H. Either the interior or the exterior of C_v in H contains v. We assume that the interior of C_v in H contains v; the other case is similar (it might be helpful to imagine that H is embedded in the sphere instead of the plane). We say that an edge e of H is *C_v-inner* if e is embedded in the interior of C_v in H and both endpoints of e are neighbors of v in H. By Statement 1 in Corollary 4.5, each C_v-inner edge must be of the form $\{v_i, f_s^2(v_i)\}$ for some $v_i \in N_H(v)$. Thus, it is obvious that there are at most $\lfloor \frac{d_H(v)}{2} \rfloor$ C_v-inner edges in H. Let S_v^{in} be the set of C_v-inner edges.

To simplify our explanation, we need several definitions. We say that an edge $\{v_i, f_s^2(v_i)\} \in E_H$ is *duplicatable*, if none of edges $\{v, v_i\}$, $\{v, f_s(v_i)\}$, and $\{v, f_s^2(v_i)\}$ crosses an edge in H. Obviously, each duplicatable edge is not a C_v-inner edge. Moreover, for each duplicatable edge $e = \{v_i, f_s^2(v_i)\} \in E_H$, H remains to be a 1-plane embedding (of a 1-planar multigraph) even if we modify H by making a copy e_c of e and embedding e_c in H in such a way that e_c crosses only edge $\{v, f_s(v_i)\}$ in H. We say that two duplicatable edges $\{v_i, f_s^2(v_i)\}$ and $\{v_j, f_s^2(v_j)\}$ *conflict* if $v_j = f_s(v_i)$ or $v_i = f_s(v_j)$. Intuitively, if two duplicatable edges $e = \{v_i, f_s^2(v_i)\}$ and $e' = \{v_j, f_s^2(v_j)\}$ conflict, then after we use e to modify H as above, e' will be no longer duplicatable in the modified H. On the other hand, if S is a set of duplicatable edges in which no two conflict, then H remains to be a 1-plane embedding even if we use the edges in S one after another to modify H as above.

Throughout the rest of this section, fix a maximal set S_v^{du} of duplicatable edges in which no two edges conflict. Obviously, S_v^{du} can be computed in $O(1)$ time.

Fact 5.1 *Suppose that we obtain a 1-plane embedding K_v (of a 1-planar multigraph) by modifying H as follows: For each edge $e = \{v_i, f_s^2(v_i)\} \in S_v^{\text{du}}$, make a copy e_c of e and embed e_c in H in such a way that e_c crosses only edge $\{v, f_s(v_i)\}$ in H. Then, $|S_v^{\text{in}}| + |S_v^{\text{du}}|$ is also the number of edges $\{v, v_j\}$ with $1 \le j \le k$ such that edge $\{v, v_j\}$ crosses another edge in K_v.*

PROOF. Let $\#_v$ be the number of edges $\{v, v_j\}$ with $1 \le j \le k$ such that edge $\{v, v_j\}$ crosses another edge in K_v. Obviously, $|S_v^{\text{in}}| + |S_v^{\text{du}}|$ bounds $\#_v$ from below. By Condition 3 in Lemma 4.2, $|S_v^{\text{in}}| + |S_v^{\text{du}}|$ bounds $\#_v$ from above. Thus, $|S_v^{\text{in}}| + |S_v^{\text{du}}| = \#_v$. □

Since K_v is a 1-plane embedding, it is easy to see that $|S_v^{\text{in}}| + |S_v^{\text{du}}| \le 3$ if $d_H(v) = 7$, while $|S_v^{\text{in}}| + |S_v^{\text{du}}| \le 4$ if $d_H(v) = 8$.

Note that our algorithm does not actually construct K_v; we rather use K_v here only to simplify our explanation.

5.1 The Case Where $d_H(v) = 7$

Recall that $|S_v^{\text{in}}| + |S_v^{\text{du}}| \le 3$ for $d_H(v) = 7$. First, assume that $|S_v^{\text{in}}| + |S_v^{\text{du}}| \le 2$. Then, by Fact 5.1, it is easy to see that there is some $v_i \in N_H(v)$ such that none of the three edges $\{v, f_p(v_i)\}$, $\{v, v_i\}$, and $\{v, f_s(v_i)\}$ crosses an edge in K_v. Thus, by the maximality of S_v^{du}, $\{f_p(v_i), f_s(v_i)\} \notin E_H$. In turn, it is obvious that $(f_p(v_i), f_s(v_i))$ is a mergable pair for v if $f_p(v_i)$ is small, while $(f_s(v_i), f_p(v_i))$ is a mergable pair for v otherwise.

Next, assume that $|S_v^{\rm in}| + |S_v^{\rm du}| = 3$. Then, by a simple inspection, there is a $v_i \in N_H(v)$ such that $S_v^{\rm in} \cup S_v^{\rm du} = \{\{v_i, f_s^2(v_i)\}, \{f_s^2(v_i), f_s^4(v_i)\}, \{f_s^4(v_i), f_s^6(v_i)\}\}$. By a relabeling if necessary, we can assume that $v_i = v_5$ (see Figure 7(1)).

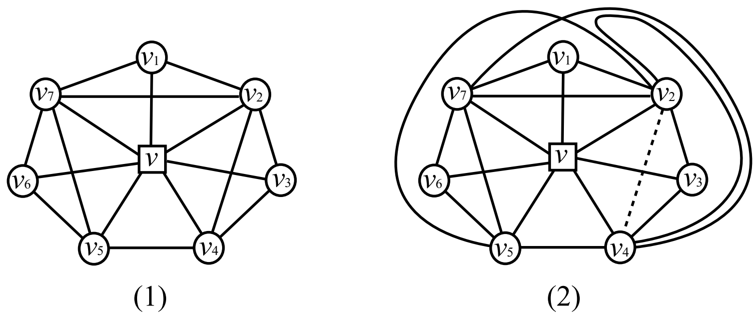


Figure 7: (1) Vertex v and its neighbors in K_v when $|S_v^{\rm in}|+|S_v^{\rm du}| = 3$. (2) Modifying K_v as follows: If $\{v_2, v_4\} \notin S_v^{\rm in}$, then delete the copy (in the interior of C_v) of edge $\{v_2, v_4\}$; otherwise, move edge $\{v_2, v_4\} \in S_v^{\rm in}$ to the exterior of C_v in such a way that edge $\{v_2, v_4\} \in S_v^{\rm in}$ is drawn as close to edges $\{v_5, v_2\}$ and $\{v_7, v_4\}$ as possible so that it crosses no edge.

If $\{v_5, v_2\} \notin E_H$, then obviously (v_5, v_2) (respectively, (v_2, v_5)) is a mergable pair for v when v_5 (respectively, v_2) is small. Similarly, if $\{v_7, v_4\} \notin E_H$, then obviously (v_7, v_4) (respectively, (v_4, v_7)) is a mergable pair for v when v_7 (respectively, v_4) is small. On the other hand, if both $\{v_5, v_2\} \in E_H$ and $\{v_7, v_4\} \in E_H$, then no matter whether edge $\{v_2, v_4\} \in S_v^{\rm in}$ or not, we can modify K_v so that K_v remains to be a 1-plane embedding and edge $\{v, v_3\}$ crosses no edge in K_v (see Figure 7(2)). As can be seen from Figure 7(2), the original embedding witnesses that $\{v_5, v_3\} \notin E_H$, and the modified embedding witnesses that (v_3, v_5) (respectively, (v_5, v_3)) is a mergable pair for v when v_3 (respectively, v_5) is small.

5.2 The Case Where $d_H(v) = 8$ and $|S_v^{\rm in}| + |S_v^{\rm du}| \leq 2$

First, assume that $|S_v^{\rm in}| + |S_v^{\rm du}| \leq 1$. Then, by Fact 5.1, it is easy to see that there are two distinct vertices v_i and v_j in $N_H(v)$ such that the six edges $\{v, f_p(v_i)\}$, $\{v, v_i\}$, $\{v, f_s(v_i)\}$, $\{v, f_p(v_j)\}$, $\{v, v_j\}$, and $\{v, f_s(v_j)\}$ are distinct and none of them crosses an edge in K_v. Thus, by the maximality of $S_v^{\rm du}$, $\{f_p(v_i), f_s(v_i)\} \notin E_H$ and $\{f_p(v_j), f_s(v_j)\} \notin E_H$. In turn, it is obvious that $(f_p(v_i), f_s(v_i))$ and $(f_p(v_j), f_s(v_j))$ are two simultaneously mergable pairs for v.

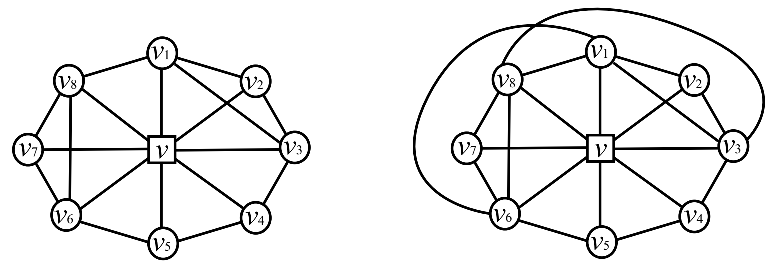
Next, assume that $|S_v^{\rm in}| + |S_v^{\rm du}| = 2$. Then, one of the following three cases occurs.

Case 1: There is a vertex $v_i \in N_H(v)$ such that $S_v^{\rm in} \cup S_v^{\rm du} = \{\{v_i, f_s^2(v_i)\}, \{f_s^4(v_i), f_s^6(v_i)\}\}$. Then, since $|S_v^{\rm in}| + |S_v^{\rm du}| = 2$, none of the six edges $\{v, v_h\}$ with $v_h \in N_H(v) - \{f_s(v_i), f_s^5(v_i)\}$ crosses an edge in K_v. Thus, by the maximality of $S_v^{\rm du}$, neither $\{f_s^2(v_i), f_s^4(v_i)\}$ nor $\{f_s^6(v_i), v_i\}$ is an edge of H, and $(f_s^2(v_i), f_s^4(v_i))$ and $(f_s^6(v_i), v_i)$ are two simultaneously mergable pairs for v.

Case 2: There is a vertex $v_i \in N_H(v)$ such that $S_v^{\rm in} \cup S_v^{\rm du} = \{\{v_i, f_s^2(v_i)\}, \{f_s^3(v_i), f_s^5(v_i)\}\}$. By a relabeling if necessary, we can assume that $v_i = v_6$ (see Figure 8(1)).

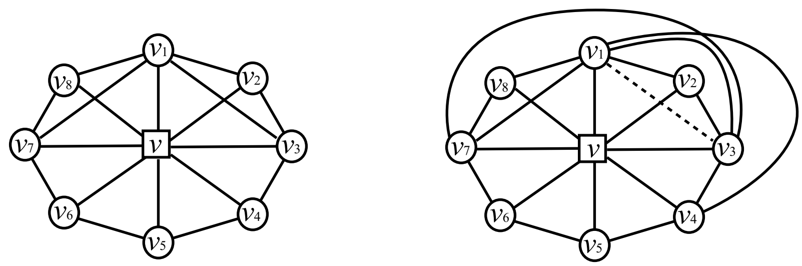
Since $|S_v^{\rm in}| + |S_v^{\rm du}| = 2$, none of the six edges $\{v, v_h\}$ with $v_h \in N_H(v) - \{v_7, v_2\}$ crosses an edge in H. Thus, by the maximality of $S_v^{\rm du}$, $\{v_3, v_5\} \notin E_H$ and $\{v_4, v_6\} \notin E_H$. Consequently, if $\{v_6, v_1\} \notin E_H$, then as can be seen from Figure 8(1), (v_3, v_5) and (v_6, v_1) are two simultaneously mergable pairs for v. Similarly, if $\{v_8, v_3\} \notin E_H$, then as can be seen from Figure 8(1), (v_4, v_6) and (v_8, v_3) are two simultaneously mergable pairs for v. On the other hand, if both $\{v_6, v_1\} \in E_H$ and $\{v_8, v_3\} \in E_H$, then as can be seen from Figure 8(2), neither $\{v_8, v_5\}$ nor $\{v_1, v_4\}$ can be an edge of H, and (v_8, v_5) and (v_1, v_4) are two simultaneously mergable pairs for v.

Case 3: There is a vertex $v_i \in N_H(v)$ such that $S_v^{\rm in} \cup S_v^{\rm du} = \{\{v_i, f_s^2(v_i)\}, \{f_s^2(v_i), f_s^4(v_i)\}\}$. By a relabeling if necessary, we can assume that $v_i = v_7$ (see Figure 9(1)).



(1) (2)

Figure 8: (1) Vertex v and its neighbors in K_v when $S_v^{\text{in}} \cup S_v^{\text{du}} = \{\{v_8, v_6\}, \{v_1, v_3\}\}$. (2) Same as (1), but in addition $\{v_6, v_1\} \in E_H$ and $\{v_8, v_3\} \in E_H$.



(1) (2)

Figure 9: (1) Vertex v and its neighbors in K_v when $S_v^{\text{in}} \cup S_v^{\text{du}} = \{\{v_1, v_7\}, \{v_1, v_3\}\}$. (2) Modifying K_v as follows: If $\{v_1, v_3\} \notin S_v^{\text{in}}$, then delete the copy (in the interior of C_c) of edge $\{v_1, v_3\}$; otherwise, move edge $\{v_1, v_3\} \in S_v^{\text{in}}$ to the exterior of C_v in such a way that edge $\{v_1, v_3\} \in S_v^{\text{in}}$ is drawn as close to edges $\{v_7, v_3\}$ and $\{v_1, v_4\}$ as possible so that it crosses no edge.

Since $|S_v^{\text{in}}| + |S_v^{\text{du}}| = 2$, none of the six edges $\{v, v_h\}$ with $v_h \in N_H(v) - \{v_8, v_2\}$ crosses an edge in H. Thus, by the maximality of S_v^{du}, $\{v_3, v_5\} \notin E_H$ and $\{v_5, v_7\} \notin E_H$. Consequently, if $\{v_7, v_3\} \notin E_H$, then as can be seen from Figure 9(1), $\{v_7, v_3, v_5\}$ is a mergable triple for v. Similarly, if $\{v_1, v_4\} \notin E_H$, then as can be seen from Figure 9(1), (v_5, v_7) and (v_1, v_4) are two simultaneously mergable pairs for v. On the other hand, if both $\{v_7, v_3\} \in E_H$ and $\{v_1, v_4\} \in E_H$, then no matter whether edge $\{v_1, v_3\} \in S_v^{\text{in}}$ or not, we can modify K_v so that K_v remains to be a 1-plane embedding and edge $\{v, v_2\}$ crosses no edge in K_v (see Figure 9(2)). As can be seen from Figure 9(2), $\{v_2, v_5, v_7\}$ must be an independent set of H and the modified embedding witnesses that $\{v_2, v_5, v_7\}$ is a mergable triple for v.

5.3 The Case Where $d_H(v) = 8$ and $|S_v^{\text{in}}| + |S_v^{\text{du}}| = 3$

This case is very complicated. Since $|S_v^{\text{in}}| + |S_v^{\text{du}}| = 3$, Fact 5.1 guarantees that one of the following cases occurs:

1. There is a vertex $v_i \in N_H(v)$ with $S_v^{\text{in}} \cup S_v^{\text{du}} = \{\{v_i, f_s^2(v_i)\}, \{f_s^2(v_i), f_s^4(v_i)\}, \{f_s^5(v_i), f_s^7(v_i)\}\}$. By a relabeling if necessary, we can assume that $v_i = v_7$ (see Figure 10(1)).

2. There is a vertex $v_i \in N_H(v)$ with $S_v^{\text{in}} \cup S_v^{\text{du}} = \{\{v_i, f_s^2(v_i)\}, \{f_s^2(v_i), f_s^4(v_i)\}, \{f_s^4(v_i), f_s^6(v_i)\}\}$. By a relabeling if necessary, we can assume that $v_i = v_6$ (see Figure 10(2)).

Case 1: $S_v^{\text{in}} \cup S_v^{\text{du}} = \{\{v_i, f_s^2(v_i)\}, \{f_s^2(v_i), f_s^4(v_i)\}, \{f_s^5(v_i), f_s^7(v_i)\}\}$ for some $v_i \in N_H(v)$. We may assume that $v_i = v_7$ (see Figure 10(1)). Consider the following four possible edges: $e_1 = \{v_1, v_4\}$, $e_2 = \{v_3, v_6\}$, $e_3 = \{v_4, v_7\}$, and $e_4 = \{v_1, v_6\}$. Each of these edges may or may not be an edge of H. So, we illustrate them by broken edges in Figure 11.

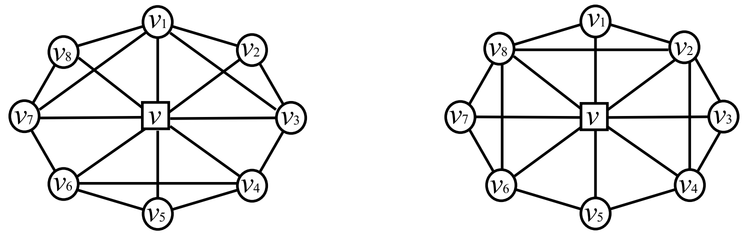


Figure 10: (1) Vertex v and its neighbors in K_v when $S_v^{\rm in} \cup S_v^{\rm du} = \{\{v_7, v_1\}, \{v_1, v_3\}, \{v_4, v_6\}\}$. (2) Vertex v and its neighbors in K_v when $S_v^{\rm in} \cup S_v^{\rm du} = \{\{v_6, v_8\}, \{v_8, v_2\}, \{v_2, v_4\}\}$.

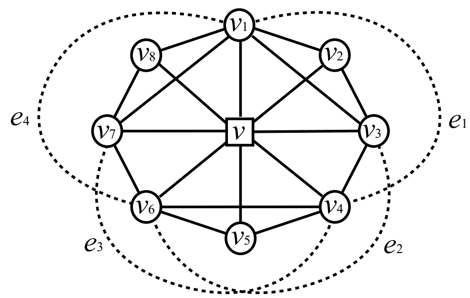


Figure 11: The four possible edges $e_1, \ldots, e_4$ when $S_v^{\rm in} \cup S_v^{\rm du} = \{\{v_7, v_1\}, \{v_1, v_3\}, \{v_4, v_6\}\}$.

If both $e_1 \in E_H$ and $e_2 \in E_H$, then no matter whether edge $\{v_1, v_3\} \in S_v^{\rm in}$ or not, we can modify K_v so that K_v remains to be a 1-plane embedding and edge $\{v, v_2\}$ crosses no edge in K_v (see Figure 12). As can be seen from Figure 12, the original embedding witnesses that $\{v_2, v_4, v_7\}$ is an independent set of H, and the modified embedding witnesses that $\{v_2, v_4, v_7\}$ is a mergable triple for v. Similarly (by symmetry), if both $e_3 \in E_H$ and $e_4 \in E_H$, then $\{v_3, v_6, v_8\}$ is a mergable triple for v.

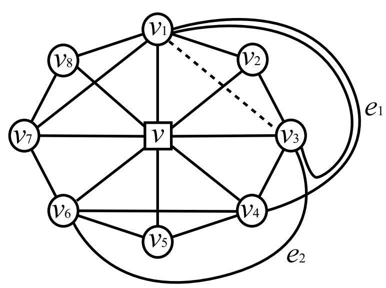


Figure 12: Vertex v and its neighbors in K_v when $S_v^{\rm in} \cup S_v^{\rm du} = \{\{v_7, v_1\}, \{v_1, v_3\}, \{v_4, v_6\}\}$ and $\{e_1, e_2\} \subseteq E_H$. In this case, we modify K_v as follows: If $\{v_1, v_3\} \not\in S_v^{\rm in}$, then delete the copy (in the interior of C_v) of edge $\{v_1, v_3\}$; otherwise, move edge $\{v_1, v_3\} \in S_v^{\rm in}$ to the exterior of C_v in such a way that edge $\{v_1, v_3\} \in S_v^{\rm in}$ is drawn as close to edges e_1 and e_2 as possible so that it crosses no edge.

Thus, it remains to consider those cases where $|\{e_1, e_2\} \cap E_H| \le 1$ and $|\{e_3, e_4\} \cap E_H| \le 1$. To this end, we further distinguish five cases (one of them must occur) as follows.

Case 1.1: $\{e_1, \ldots, e_4\} \cap E_H = \{e_2, e_3\}$. Then, no matter whether edge $\{v_4, v_6\} \in S_v^{\rm in}$ or not, we can modify K_v so that K_v remains to be a 1-plane embedding and edge $\{v, v_5\}$ crosses no edge in K_v (see Figure 13). As can be seen from Figure 13, the original embedding witnesses that $\{v_1, v_6\} \not\in E_H$ and $\{v_3, v_5\} \not\in E_H$, and the modified embedding witnesses that (v_1, v_6) and (v_3, v_5) are two simultaneously mergable pairs for v.

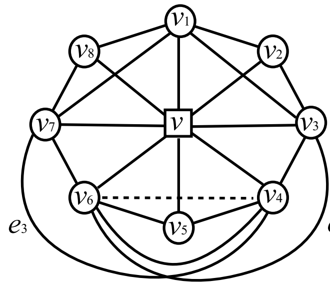


Figure 13: Vertex v and its neighbors in K_v when $S_v^{\text{in}} \cup S_v^{\text{du}} = \{\{v_7, v_1\}, \{v_1, v_3\}, \{v_4, v_6\}\}$ and $\{e_1, \ldots, e_4\} \cap E_H = \{e_2, e_3\}$. In this case, we modify K_v as follows: If $\{v_4, v_6\} \notin S_v^{\text{in}}$, then delete the copy (in the interior of C_v) of edge $\{v_4, v_6\}$; otherwise, move edge $\{v_4, v_6\} \in S_v^{\text{in}}$ to the exterior of C_v in such a way that edge $\{v_4, v_6\} \in S_v^{\text{in}}$ is drawn as close to edges e_2 and e_3 as possible so that it crosses no edge.

Case 1.2: $\{e_1, \ldots, e_4\} \cap E_H = \{e_1, e_4\}$. If $d_H(v_1) \leq 8$, then as can be seen from Figure 14(1), $\{v_2, v_8\} \notin E_H$, $\{v_3, v_7\} \notin E_H$, and $(v_1, v_2, v_8, v_3, v_7)$ is a favorite quintuple for v. So, assume that $d_H(v_1) \geq 9$. Then, by Figure 10(1) and Condition 3 in the definition of reducibility of v in Section 4.1, at least one of v_4 and v_6 has degree ≤ 8 in H. By symmetry, we assume that $d_H(v_4) \leq 8$ (see Figure 14(2)). By Figure 14(2), $\{v_3, v_5\} \notin E_H$ or else $d_H(v_4)$ would have been 5 by the 3-connectivity of H, contradicting the criticality of v. Similarly (by symmetry), $\{v_5, v_7\} \notin E_H$. Thus, $\{v_3, v_5, v_7\}$ is an independent set of H and (v_4, v_3, v_5, v_7) is a desired quadruple for v.

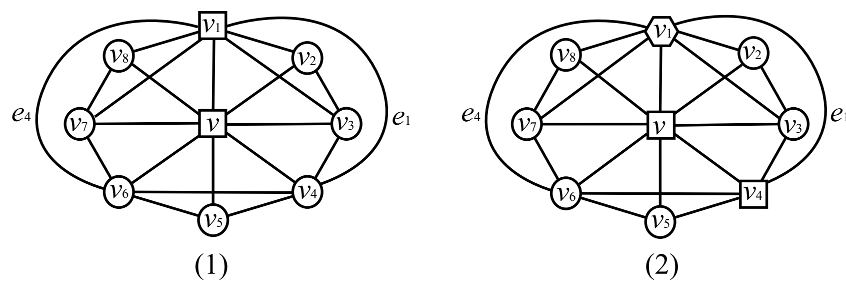
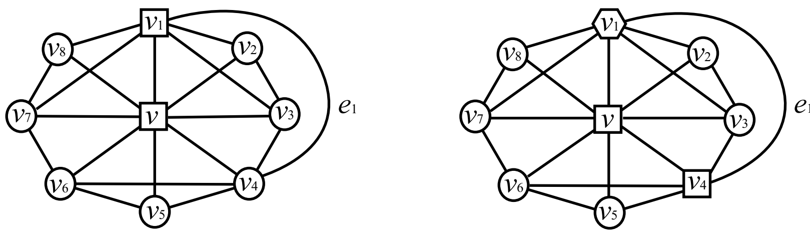


Figure 14: (1) Vertex v and its neighbors in K_v when $S_v^{\text{in}} \cup S_v^{\text{du}} = \{\{v_7, v_1\}, \{v_1, v_3\}, \{v_4, v_6\}\}$, $\{e_1, \ldots, e_4\} \cap E_H = \{e_1, e_4\}$, and $d_H(v_1) \leq 8$. (2) Vertex v and its neighbors in K_v when $S_v^{\text{in}} \cup S_v^{\text{du}} = \{\{v_7, v_1\}, \{v_1, v_3\}, \{v_4, v_6\}\}$, $\{e_1, \ldots, e_4\} \cap E_H = \{e_1, e_4\}$, $d_H(v_1) \geq 9$, and $d_H(v_4) \leq 8$.

Case 1.3: Either $\{e_1, e_2, e_4\} \cap E_H = \{e_1\}$ or $\{e_4, e_1, e_3\} \cap E_H = \{e_4\}$. By symmetry, we assume that $\{e_1, e_2, e_4\} \cap E_H = \{e_1\}$. Note that e_3 may be an edge of H or may be not.

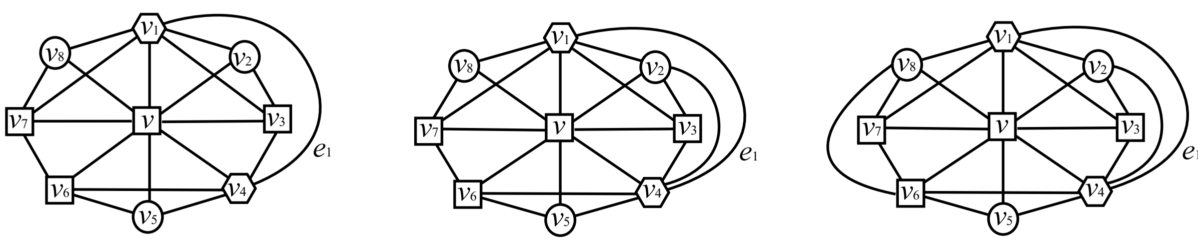
If $d_H(v_1) \leq 8$, then as can be seen from Figure 15(1), $\{v_2, v_8\} \notin E_H$ (or else $d_H(v_1)$ would have been 6 by the 3-connectivity of H, contradicting the criticality of v), and in turn $(v_1, v_2, v_8, v_3, v_6)$ is a favorite quintuple for v. So, assume that $d_H(v_1) \geq 9$. If $d_H(v_4) \leq 8$, then as can be seen from Figure 15(2), $\{v_3, v_5\} \notin E_H$ (or else $d_H(v_4)$ would have been 5 by the 3-connectivity of H, contradicting the criticality of v), and in turn $(v_4, v_3, v_5, v_1, v_6)$ is a favorite quintuple for v. So, further assume that $d_H(v_4) \geq 9$. Then, by Figure 10(1) and Condition 3 in the definition of reducibility of v in Section 4.1, each of v_3, v_6, and v_7 has degree ≤ 8 in H (see Figure 16(1)). If $\{v_2, v_4\} \notin E_H$, then as can be seen from Figure 16(1), $(v_3, v_2, v_4, v_6, v_1)$ is a favorite quintuple for v. So, yet further assume that $\{v_2, v_4\} \in E_H$. If in addition $\{v_6, v_8\} \notin E_H$, then as can be seen from Figure 16(2), $\{v_3, v_6, v_8\}$ must be an independent set of H and in turn (v_7, v_6, v_8, v_3) is a desired quadruple for v. On the other hand, if $\{v_6, v_8\} \in E_H$, then as can be seen from Figure 16(3),



(1) (2)

Figure 15: (1) Vertex v and its neighbors in K_v when $S_v^{\text{in}} \cup S_v^{\text{du}} = \{\{v_7, v_1\}, \{v_1, v_3\}, \{v_4, v_6\}\}$, $\{e_1, e_2, e_4\} \cap E_H = \{e_1\}$, and $d_H(v_1) \le 8$. (2) Vertex v and its neighbors in K_v when $S_v^{\text{in}} \cup S_v^{\text{du}} = \{\{v_7, v_1\}, \{v_1, v_3\}, \{v_4, v_6\}\}$, $\{e_1, e_2, e_4\} \cap E_H = \{e_1\}$, $d_H(v_1) \ge 9$, and $d_H(v_4) \le 8$.

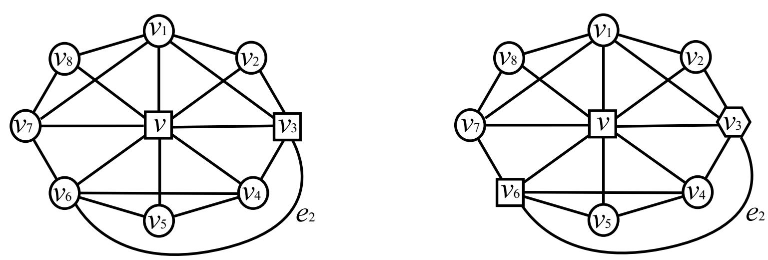
$\{v_5, v_7\} \notin E_H$ (or else $d_H(v_6)$ would have been 5 by the 3-connectivity of H, contradicting the criticality of v), and in turn $\{v_3, v_5, v_7\}$ is an independent set of H and (v_6, v_5, v_7, v_3) is a desired quadruple for v.



(1) (2) (3)

Figure 16: (1) Vertex v and its neighbors in K_v when $S_v^{\text{in}} \cup S_v^{\text{du}} = \{\{v_7, v_1\}, \{v_1, v_3\}, \{v_4, v_6\}\}$, $\{e_1, e_2, e_4\} \cap E_H = \{e_1\}$, $d_H(v_1) \ge 9$, and $d_H(v_4) \ge 9$. (2) Same as (1), but in addition $\{v_2, v_4\} \in E_H$. (3) Same as (2), but in addition $\{v_6, v_8\} \in E_H$.

Case 1.4: Either $\{e_1, \dots, e_4\} \cap E_H = \{e_2\}$ or $\{e_1, \dots, e_4\} \cap E_H = \{e_3\}$. By symmetry, we assume that $\{e_1, \dots, e_4\} \cap E_H = \{e_2\}$.



(1) (2)

Figure 17: (1) Vertex v and its neighbors in K_v when $S_v^{\text{in}} \cup S_v^{\text{du}} = \{\{v_7, v_1\}, \{v_1, v_3\}, \{v_4, v_6\}\}$, $\{e_1, \dots, e_4\} \cap E_H = \{e_2\}$, and $d_H(v_3) \le 8$. (2) Vertex v and its neighbors in K_v when $S_v^{\text{in}} \cup S_v^{\text{du}} = \{\{v_7, v_1\}, \{v_1, v_3\}, \{v_4, v_6\}\}$, $\{e_1, \dots, e_4\} \cap E_H = \{e_2\}$, $d_H(v_3) \ge 9$, and $d_H(v_6) \le 8$.

If $d_H(v_3) \le 8$, then as can be seen from Figure 17(1), $\{v_2, v_4\} \notin E_H$ (or else $d_H(v_3)$ would have been 5 by the 3-connectivity of H, contradicting the criticality of v), and in turn $(v_3, v_2, v_4, v_6, v_1)$ is a favorite quintuple for v. So, assume that $d_H(v_3) \ge 9$. If $d_H(v_6) \le 8$, then as can be seen from Figure 17(2), $\{v_5, v_7\} \notin E_H$ (or else $d_H(v_6)$ would have been 5 by the 3-connectivity of H, contradicting the criticality of v), and in turn $(v_6, v_5, v_7, v_1, v_4)$ is a favorite quintuple for v. So, further assume that $d_H(v_6) \ge 9$. Then, by Figure 10(1) and Condition 3 in the definition of reducibility of v in Section 4.1, each of v_1, v_4, v_7, and v_8 has degree ≤ 8 in H. If $\{v_8, v_3\} \notin E_H$, then as can be seen from Figure 10(1), $(v_1, v_8, v_3, v_4, v_7)$ is a favorite quintuple for v. So, yet further assume that $\{v_8, v_3\} \in E_H$ (see Figure 18(1)). If in addition $\{v_2, v_7\} \notin E_H$, then as can be seen from Figure 18(1), $\{v_2, v_4, v_7\}$ must be an independent set of H and in turn (v_1, v_2, v_7, v_4) is a desired quadruple for v. On the other hand, if $\{v_2, v_7\} \in E_H$, then as can be seen from Figure 18(2), $\{v_6, v_8\} \notin E_H$ and in turn $(v_7, v_6, v_8, v_1, v_4)$ is a favorite quintuple for v.

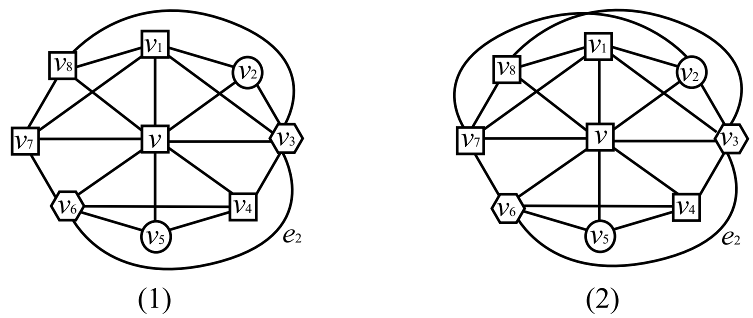


Figure 18: (1) Vertex v and its neighbors in K_v when $S_v^{\text{in}} \cup S_v^{\text{du}} = \{\{v_7, v_1\}, \{v_1, v_3\}, \{v_4, v_6\}\}$, $\{e_1, \ldots, e_4\} \cap E_H = \{e_2\}$, $d_H(v_3) \geq 9$, $d_H(v_6) \geq 9$, and $\{v_3, v_8\} \in E_H$. (2) Same as (1), but in addition $\{v_2, v_7\} \in E_H$.

Case 1.5: $\{e_1, \ldots, e_4\} \cap E_H = \emptyset$. We yet further distinguish two cases as follows.

Case 1.5.1: $d_H(v_1) \leq 8$. If $\{v_2, v_7\} \notin E_H$, then as can be seen from Figure 10(1), $(v_1, v_2, v_7, v_3, v_6)$ is a favorite quintuple for v. Similarly, if $\{v_3, v_8\} \notin E_H$, then $(v_1, v_3, v_8, v_4, v_7)$ is a favorite quintuple for v. So, assume that both $\{v_2, v_7\} \in E_H$ and $\{v_3, v_8\} \in E_H$ (see Figure 19(1)). Note that both $\{v_2, v_8\} \in E_H$ and $\{v_3, v_7\} \in E_H$ by Condition 3 in Lemma 4.2.

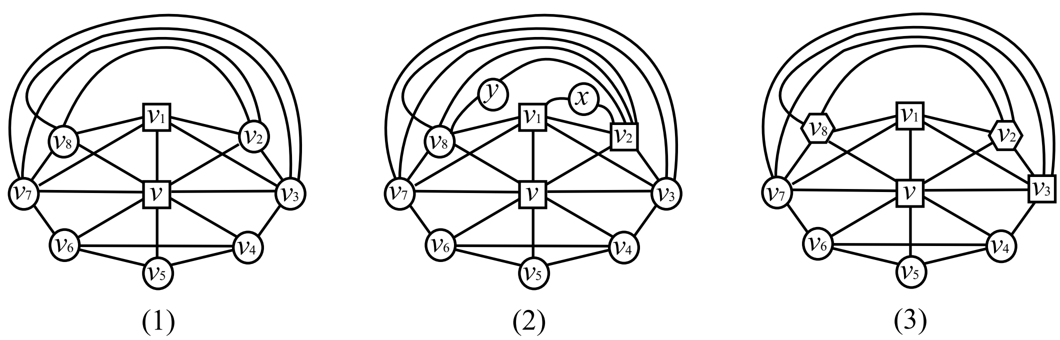


Figure 19: (1) Vertex v and its neighbors in K_v when $S_v^{\text{in}} \cup S_v^{\text{du}} = \{\{v_7, v_1\}, \{v_1, v_3\}, \{v_4, v_6\}\}$, $\{e_1, \ldots, e_4\} \cap E_H = \emptyset$, $d_H(v_1) \leq 8$, $\{v_3, v_8\} \in E_H$, and $\{v_2, v_7\} \in E_H$. (2) Same as (1), but in addition $d_H(v_2) \leq 8$. (3) Vertex v and its neighbors in K_v when $S_v^{\text{in}} \cup S_v^{\text{du}} = \{\{v_7, v_1\}, \{v_1, v_3\}, \{v_4, v_6\}\}$, $\{e_1, \ldots, e_4\} \cap E_H = \emptyset$, $d_H(v_1) \leq 8$, $d_H(v_2) \geq 9$, $d_H(v_8) \geq 9$, $d_H(v_3) \leq 8$, $\{v_3, v_8\} \in E_H$, and $\{v_2, v_7\} \in E_H$.

First, suppose that $d_H(v_2) \leq 8$ or $d_H(v_8) \leq 8$. By symmetry in Figure 19(1), we assume that $d_H(v_2) \leq 8$. Then (see Figure 19(2)), since $d_H(v_2) \geq 7$ by the criticality of v, we can find the vertices $x \in N_H(v_2)$ and $y \in N_H(v_2)$ such that v_1 and x appear around v_2 consecutively in H in this order clockwise, and v_8 and y appear around v_2 consecutively in H in this order counterclockwise. Note that both $\{v_1, x\} \in E_H$ and $\{v_8, y\} \in E_H$ by Condition 2 in Lemma 4.2. If $d_H(v_2) = 7$, then x and y are consecutive neighbors of v_2 in H and hence at least one of edges $\{v_2, x\}$ and $\{v_2, y\}$ crosses no edge in H by Statement 2 in Corollary 4.5. Moreover, if $d_H(v_2) = 7$ and edge $\{v_2, x\}$ crosses no edge in H, then as can be seen from Figure 19(2), (v_2, v_3, x) is a useful triple for v. Similarly, if $d_H(v_2) = 7$ and edge $\{v_2, y\}$ crosses no edge in H, then (v_2, v_3, y) is a useful triple for v. So, we are done when $d_H(v_2) = 7$. Thus, assume that $d_H(v_2) = 8$. Let z be the vertex in $N_H(v_2) - \{x, v_1, v, v_3, v_7, v_8, y\}$. By the 3-connectivity of H and our choice of x and y, vertices x, z, y appear around v_2 consecutively in H in this order clockwise. In turn, by Statement 2 in Corollary 4.5, at least one of edges $\{v_2, y\}$ and $\{v_2, z\}$ crosses no edge in H. If $\{v_2, y\}$ crosses no edge in H, then as can be seen from Figure 19(2), H remains to be a 1-plane embedding even if we delete v_1 and v_2, merge x and v along the two original edges $\{x, v_1\}$ and $\{v_1, v\}$, and merge y and v_3 along the two original edges $\{y, v_2\}$ and $\{v_2, v_3\}$; so, (v_2, v_1, v, x, v_3, y) is a useful sextuple for v. Similarly, if $\{v_2, z\}$ crosses no edge in H, then (v_2, v_1, v, x, v_3, z) is a useful sextuple for v.

Next, suppose that both $d_H(v_2) \geq 9$ and $d_H(v_8) \geq 9$. Then as can be seen from Figure 10(1), Condition 3 in the definition of the reducibility of v guarantees that $d_H(v_3) \leq 8$ or $d_H(v_7) \leq 8$. If $d_H(v_3) \leq 8$, then as can be seen Figure 19(3), $(v_3, v_2, v_4, v_1, v_6)$ is a favorite quintuple for v. Similarly, if $d_H(v_7) \leq 8$, then $(v_7, v_6, v_8, v_1, v_4)$ is a favorite quintuple for v.

Case 1.5.2: $d_H(v_1) \geq 9$. Then, as can be seen from Figure 10(1), Condition 3 in the definition of the reducibility of v guarantees that $d_H(v_3) \leq 8$ or $d_H(v_7) \leq 8$. By symmetry in Figure 10(1), we assume that $d_H(v_3) \leq 8$. If $\{v_2, v_4\} \notin E_H$, then as can be seen from Figure 10(1), $(v_3, v_2, v_4, v_1, v_6)$ is a favorite quintuple for v. So, further assume that $\{v_2, v_4\} \in E_H$. If $d_H(v_4) \leq 8$, then as can be seen from Figure 20(1), $\{v_3, v_5\} \notin E_H$ (or else $d_H(v_4)$ would have been 5 by the 3-connectivity of H, contradicting the criticality of v), and in turn $(v_4, v_3, v_5, v_1, v_6)$ is a favorite quintuple for v. So, yet further assume that $d_H(v_4) \geq 9$. Then, as can be seen from Figure 10(1), Condition 3 in the definition of the reducibility of v guarantees that $d_H(v_6) \leq 8$ and $d_H(v_7) \leq 8$ (see Figure 20(2)). If $\{v_6, v_8\} \notin E_H$, then as can be seen from Figure 20(2), $(v_7, v_6, v_8, v_1, v_4)$ is a favorite quintuple for v. Otherwise, as can be seen from Figure 20(3), $\{v_5, v_7\} \notin E_H$ (or else $d_H(v_6)$ would have been 5 by the 3-connectivity of H, contradicting the criticality of v), and in turn $(v_6, v_5, v_7, v_1, v_4)$ is a favorite quintuple for v.

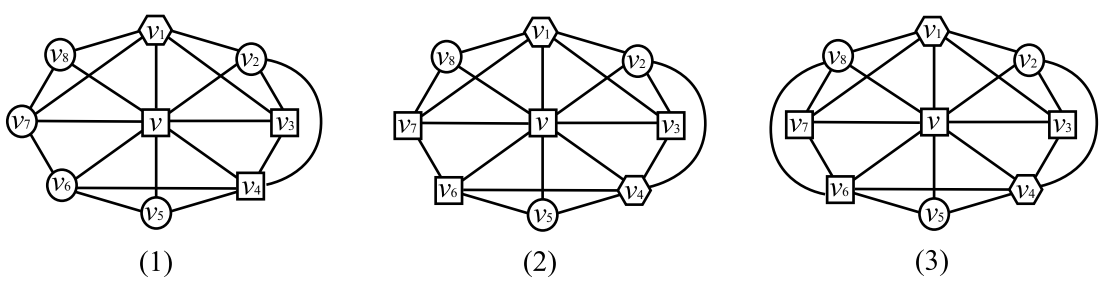


Figure 20: (1) Vertex v and its neighbors in K_v when $S_v^{\text{in}} \cup S_v^{\text{du}} = \{\{v_7, v_1\}, \{v_1, v_3\}, \{v_4, v_6\}\}$, $\{e_1, \ldots, e_4\} \cap E_H = \emptyset$, $d_H(v_1) \geq 9$, $d_H(v_3) \leq 8$, $d_H(v_4) \leq 8$, and $\{v_2, v_4\} \in E_H$. (2) Vertex v and its neighbors in K_v when $S_v^{\text{in}} \cup S_v^{\text{du}} = \{\{v_7, v_1\}, \{v_1, v_3\}, \{v_4, v_6\}\}$, $\{e_1, \ldots, e_4\} \cap E_H = \emptyset$, $d_H(v_1) \geq 9$, $d_H(v_3) \leq 8$, $d_H(v_4) \geq 9$, and $\{v_2, v_4\} \in E_H$. (3) Same as (2), but in addition $\{v_6, v_8\} \in E_H$.

Case 2: $S_v^{\text{in}} \cup S_v^{\text{du}} = \{\{v_i, f_s^2(v_i)\}, \{f_s^2(v_i), f_s^4(v_i)\}, \{f_s^4(v_i), f_s^6(v_i)\}\}$ for some $v_i \in N_H(v)$. We may assume that $v_i = v_6$ (see Figure 10(2)). Consider the following two possible edges: $e_1 = \{v_2, v_6\}$ and $e_2 = \{v_4, v_8\}$. Either edge may or may not be an edge of H. So, we illustrate them by broken edges in Figure 21.

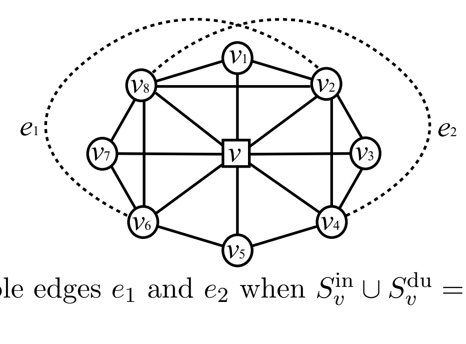


Figure 21: The two possible edges e_1 and e_2 when $S_v^{\text{in}} \cup S_v^{\text{du}} = \{\{v_6, v_8\}, \{v_8, v_2\}, \{v_2, v_4\}\}$.

We further distinguish five cases (one of them must occur) as follows.

Case 2.1: Both $e_1 \in E_H$ and $e_2 \in E_H$. Then, no matter whether edge $\{v_2, v_8\} \in S_v^{\text{in}}$ or not, we can modify K_v so that K_v remains to be a 1-plane embedding and edge $\{v, v_1\}$ crosses no edge in K_v (see Figure 22(1)). Similar modifications can be done for the other two edges $\{v_2, v_4\}$ and $\{v_6, v_8\}$ in $S_v^{\text{in}} \cup S_v^{\text{du}}$. As can be seen from Figure 22(1), the original embedding witnesses that $\{v_1, v_3, v_7\}$ is an independent set of H, and the modified embedding witnesses that $\{v_1, v_3, v_7\}$ is a mergable triple for v.

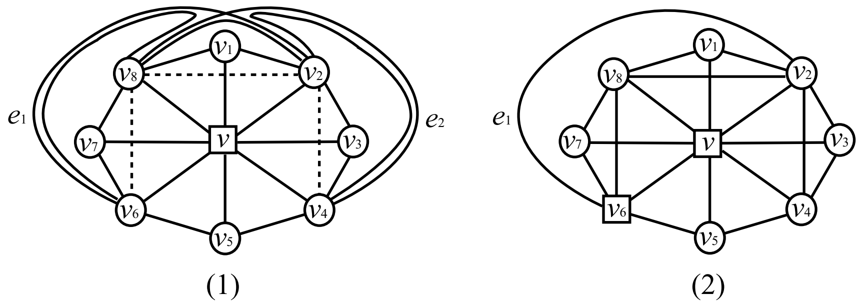


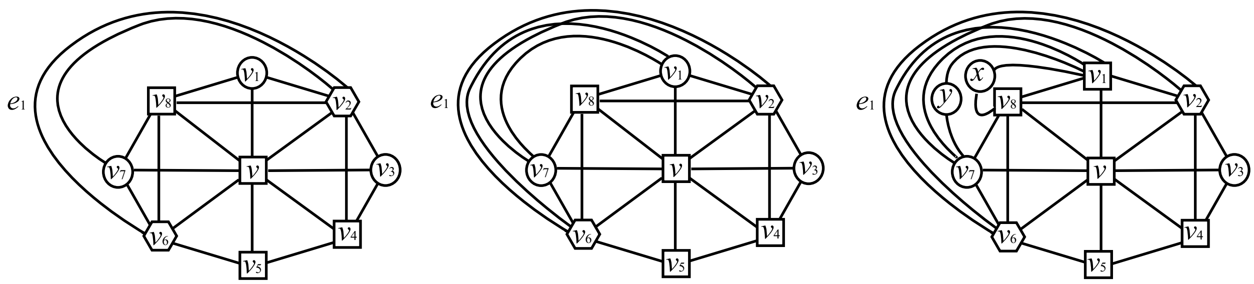
Figure 22: (1) Vertex v and its neighbors in K_v when $S_v^{\text{in}} \cup S_v^{\text{du}} = \{\{v_6, v_8\}, \{v_8, v_2\}, \{v_2, v_4\}\}$ and $\{e_1, e_2\} \subseteq E_H$. In this case, we modify K_v as follows: For each edge $e \in S_v^{\text{in}} \cup S_v^{\text{du}}$, if $e \notin S_v^{\text{in}}$, then delete the copy (in the interior of C_v) of e; otherwise, move edge $e \in S_v^{\text{in}}$ to the exterior of C_v in such a way that edge e is drawn as close to edges e_1 and e_2 as possible so that it crosses no edge. (2) Vertex v and its neighbors in K_v when $S_v^{\text{in}} \cup S_v^{\text{du}} = \{\{v_6, v_8\}, \{v_8, v_2\}, \{v_2, v_4\}\}$, $\{e_1, e_2\} \cap E_H = \{e_1\}$, and $d_H(v_6) \le 8$.

Case 2.2: Either $\{e_1, e_2\} \cap E_H = \{e_1\}$ or $\{e_1, e_2\} \cap E_H = \{e_2\}$. By symmetry in Figure 21, we assume that $\{e_1, e_2\} \cap E_H = \{e_1\}$. Depending on whether $d_H(v_2) \le 8$ or $d_H(v_6) \le 8$, we yet further distinguish two cases as follows.

Case 2.2.1: $d_H(v_2) \le 8$ or $d_H(v_6) \le 8$. If $d_H(v_6) \le 8$, then as can be seen from Figure 22(2), $\{v_5, v_7\} \notin E_H$ (or else $d_H(v_6)$ would have been 5 by the 3-connectivity of H, contradicting the criticality of v), and in turn $(v_6, v_5, v_7, v_4, v_8)$ is a favorite quintuple for v. Similarly, if $d_H(v_2) \le 8$, then $(v_2, v_1, v_3, v_4, v_8)$ is a favorite quintuple for v.

Case 2.2.2: $d_H(v_2) \ge 9$ and $d_H(v_6) \ge 9$. Then, as can be seen from Figure 10(2), Condition 3 in the definition of the reducibility of v guarantees that $d_H(v_4) \le 8$, $d_H(v_5) \le 8$, and $d_H(v_8) \le 8$. Moreover, $\{v_4, v_6\} \notin E_H$ by the maximality of S_v^{du}. Thus, if $\{v_2, v_7\} \notin E_H$, then as can be seen from Figure 10(2), $(v_8, v_2, v_7, v_4, v_6)$ is a favorite quintuple for v. So, further assume that $\{v_2, v_7\} \in E_H$. If $\{v_1, v_6\} \notin E_H$, then as can be seen from Figure 23(1), $\{v_1, v_4, v_6\}$ must be an independent set of H, and in turn (v_8, v_1, v_6, v_4) is a desired quadruple for v. So, yet further assume that $\{v_1, v_6\} \in E_H$ (see Figure 23(2)). Note that $\{v_1, v_7\} \in E_H$ by Condition 3 in Lemma 4.2. Moreover, by Condition 3 in the definition of reducibility of v, $d_H(v_1) \le 8$ or $d_H(v_7) \le 8$. We assume that $d_H(v_1) \le 8$; the case where $d_H(v_7) \le 8$ is similar. Then (see Figure 23(3)), since $d_H(v_1) \ge 7$ by the criticality of v, we can find the vertices $x \in N_H(v_1)$ and $y \in N_H(v_1)$ such that v_8 and x appear around v_1 consecutively in H in this order clockwise, and v_7 and y appear around v_1 consecutively in H in this order counterclockwise. Note that both $\{v_8, x\} \in E_H$ and $\{v_7, y\} \in E_H$ by Condition 2 in Lemma 4.2.

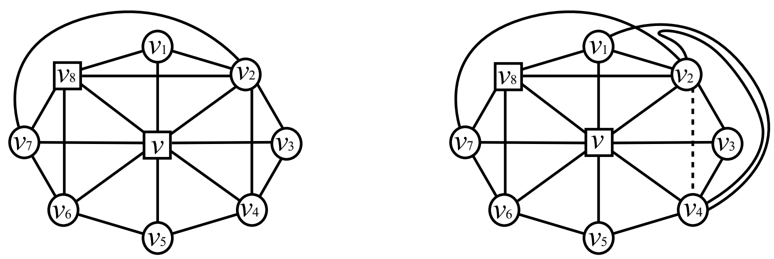
If $d_H(v_1) = 7$, then x and y are consecutive neighbors of v_1 in H and hence at least one of edges $\{v_1, x\}$ and $\{v_1, y\}$ crosses no edge in H by Statement 2 in Corollary 4.5. Moreover, if $d_H(v_1) = 7$ and edge $\{v_1, x\}$ crosses no edge in H, then as can be seen from Figure 23(3), (v_1, v_2, x) is a useful triple for v. Similarly, if $d_H(v_1) = 7$ and edge $\{v_1, y\}$ crosses no edge in H, then (v_1, v_2, y) is a useful triple for v. So, we are done when $d_H(v_1) = 7$. Thus, assume that $d_H(v_1) = 8$. Let z be the vertex in $N_H(v_1) - \{x, v_8, v, v_2, v_6, v_7, y\}$. By the 3-connectivity of H and our choice of x and y, vertices x, z, y appear around v_1 consecutively in H in this order clockwise. In turn, by Statement 2 in Corollary 4.5, at least one of edges $\{v_1, y\}$ and $\{v_1, z\}$ crosses no edge in H. If $\{v_1, y\}$ crosses no edge in H, then as can be seen from Figure 23(3), H remains to be a 1-plane embedding even if we delete v_1 and v_8, merge x and v along the two original edges $\{x, v_8\}$ and $\{v_8, v\}$, and merge y and v_2 along the two original edges $\{y, v_1\}$ and $\{v_1, v_2\}$; so, (v_1, v_8, v, x, v_2, y) is a useful sextuple for v. Similarly, if $\{v_1, z\}$ crosses no edge in H, then (v_1, v_8, v, x, v_2, z) is a useful sextuple for v.



(1) (2) (3)

Figure 23: (1) Vertex v and its neighbors in K_v when $S_v^{\text{in}} \cup S_v^{\text{du}} = \{\{v_6, v_8\}, \{v_8, v_2\}, \{v_2, v_4\}\}$, $\{e_1, e_2\} \cap E_H = \{e_1\}$, $d_H(v_2) \geq 9$, $d_H(v_6) \geq 9$, and $\{v_2, v_7\} \in E_H$. (2) Same as (1), but in addition $\{v_1, v_6\} \in E_H$. (3) Same as (2), but in addition $d_H(v_1) \leq 8$.

Case 2.3: $\{e_1, e_2\} \cap E_H = \emptyset$ and $d_H(v_8) \leq 8$. Note that $\{v_4, v_6\} \notin E_H$ by the maximality of S_v^{du}. If $\{v_2, v_7\} \notin E_H$, then as can be seen from Figure 10(2), $(v_8, v_2, v_7, v_4, v_6)$ is a favorite quintuple for v. So, assume that $\{v_2, v_7\} \in E_H$ (see Figure 24(1)). By Figure 24(1), if $\{v_1, v_6\}$ is an edge of H, then it has to cross edge $\{v_2, v_7\}$ and in turn $e_1 = \{v_2, v_6\}$ has to be an edge of H by Condition 3 in Lemma 4.2, contradicting $\{e_1, e_2\} \cap E_H = \emptyset$. Thus, $\{v_1, v_6\} \notin E_H$. Consequently, if $\{v_1, v_4\} \notin E_H$, then as can be seen from Figure 24(1), $\{v_1, v_4, v_6\}$ is an independent set of H and (v_8, v_1, v_6, v_4) is a desired quadruple for v. On the other hand, if $\{v_1, v_4\} \in E_H$, then no matter whether edge $\{v_2, v_4\} \in S_v^{\text{in}}$ or not, we can modify K_v so that K_v remains to be a 1-plane embedding and edge $\{v, v_3\}$ crosses no edge in K_v (see Figure 24(2)). As can be seen from Figure 24(2), the original embedding witnesses that $\{v_3, v_5\} \notin E_H$, and the modified embedding witnesses that (v_3, v_5) and (v_2, v_6) are two simultaneously mergable pairs for v.



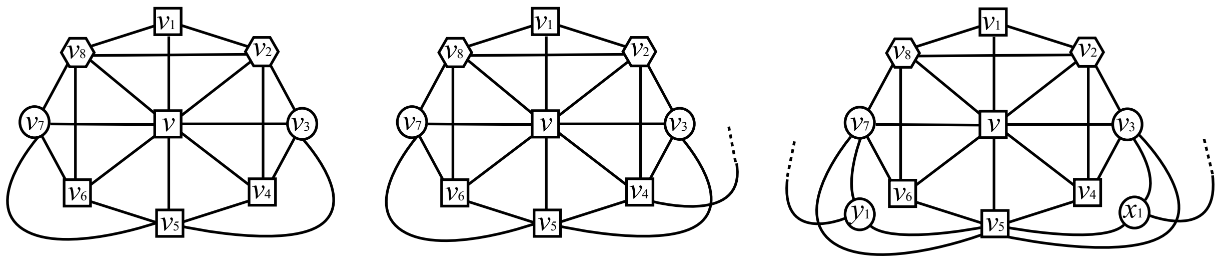
(1) (2)

Figure 24: (1) Vertex v and its neighbors in K_v when $S_v^{\text{in}} \cup S_v^{\text{du}} = \{\{v_6, v_8\}, \{v_8, v_2\}, \{v_2, v_4\}\}$, $\{e_1, e_2\} \cap E_H = \emptyset$, $d_H(v_8) \leq 8$, and $\{v_2, v_7\} \in E_H$. (2) Same as (1), but in addition $\{v_1, v_4\} \in E_H$. In this case, we modify K_v as follows: If $\{v_2, v_4\} \notin S_v^{\text{in}}$, then delete the copy (in the interior of C_v) of edge $\{v_2, v_4\}$; otherwise, move edge $\{v_2, v_4\} \in S_v^{\text{in}}$ to the exterior of C_v in such a way that edge $\{v_2, v_4\}$ is drawn as close to edges $\{v_7, v_2\}$ and $\{v_1, v_4\}$ as possible so that it crosses no edge.

Case 2.4: $\{e_1, e_2\} \cap E_H = \emptyset$ and $d_H(v_2) \leq 8$. Similar to Case 2.3 (by symmetry in Figure 10(2)).

Case 2.5: $\{e_1, e_2\} \cap E_H = \emptyset$, $d_H(v_8) \geq 9$, and $d_H(v_2) \geq 9$. Then, as can be seen from Figure 10(2), Condition 3 in the definition of reducibility of v guarantees that $d_H(v_1) \leq 8$, $d_H(v_4) \leq 8$, $d_H(v_5) \leq 8$, and $d_H(v_6) \leq 8$. In turn, if $\{v_3, v_5\} \notin E_H$, then as can be seen from Figure 10(2), $(v_4, v_3, v_5, v_2, v_6)$ is a favorite quintuple for v. Similarly (by symmetry in Figure 10(2)), if $\{v_7, v_5\} \notin$

E_H, then $(v_6, v_7, v_5, v_4, v_8)$ is a favorite quintuple for v. So, we assume that both $\{v_3, v_5\} \in E_H$ and $\{v_7, v_5\} \in E_H$ (see Figure 25(1)).



(1) (2) (3)

Figure 25: (1) Vertex v and its neighbors in K_v when $S_v^{\text{in}} \cup S_v^{\text{du}} = \{\{v_6, v_8\}, \{v_8, v_2\}, \{v_2, v_4\}\}$, $\{e_1, e_2\} \cap E_H = \emptyset$, $d_H(v_8) \geq 9$, $d_H(v_2) \geq 9$, $\{v_3, v_5\} \in E_H$, and $\{v_7, v_5\} \in E_H$. (2) Same as (1), but in addition edge $\{v_3, v_5\}$ crosses another edge e in H such that v_4 is an endpoint of e. (3) Same as (1), but in addition edges $\{v_3, v_5\}$ and $\{v_7, v_5\}$ cross edges $\{x_1, x_2\}$ and $\{y_1, y_2\}$ in H respectively such that $v_4 \notin \{x_1, x_2\}$ and $v_6 \notin \{y_1, y_2\}$. Note that x_2 and y_2 are missing.

If edge $\{v_3, v_5\}$ crosses no edge in H, then as can be seen from Figure 25(1), H remains to be a 1-plane embedding even if we delete v and v_5, merge v_4 and v_8 along the two original edges $\{v_4, v\}$ and $\{v, v_8\}$, and merge v_3 and v_6 along the two original edges $\{v_3, v_5\}$ and $\{v_5, v_6\}$; hence, $(v_5, v_3, v_6, v_4, v_8)$ is a favorite quintuple for v. Similarly (by symmetry in Figure 25(1)), if edge $\{v_7, v_5\}$ crosses no edge in H, then $(v_5, v_7, v_4, v_2, v_6)$ is a favorite quintuple for v. Thus, we assume that edge $\{v_3, v_5\}$ crosses another edge $\{x_1, x_2\}$ in H and edge $\{v_7, v_5\}$ crosses another edge $\{y_1, y_2\}$ in H. Then, $v_4 \notin \{x_1, x_2\}$; otherwise, as can be seen from Figure 25(2), $d_H(v_4)$ would have been 5 by the 3-connectivity of H, contradicting the criticality of v. Similarly, $v_6 \notin \{y_1, y_2\}$. Figure 25(3) illustrates these facts. Vertices x_2 and y_2 are missing in Figure 25(3), because they may be the same or may be not, and they may be in $N_H(v)$ or may be not. No matter whether $x_2 = y_2$ and no matter whether $\{x_2, y_2\} \cap N_H(v) = \emptyset$, the following statements hold:

1. Both $\{v_5, x_2\} \in E_H$ and $\{v_5, y_2\} \in E_H$. This follows from Condition 3 in Lemma 4.2.

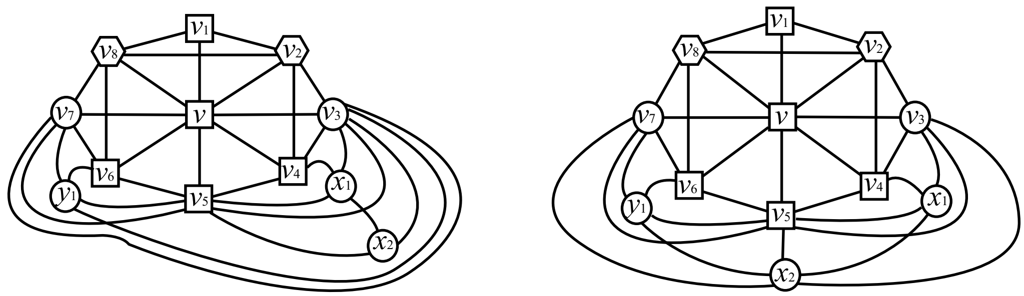
2. Vertices v, v_4, x_1, v_3, v_7, y_1, and v_6 are 7 distinct neighbors of v_5 in H. This can be seen from Figure 25(3) immediately.

Depending on whether $x_2 = y_2$ and whether $\{x_2, y_2\} \cap N_H(v) = \emptyset$, we yet further distinguish three cases (one of them must occur) as follows.

Case 2.5.1: $x_2 \neq y_2$. Then, by Statements 1 and 2 above and the fact that $d_H(v_5) \leq 8$, we have $\{x_2, y_2\} \cap \{x_1, y_1, v, v_3, v_4, v_6, v_7\} \neq \emptyset$. On the other hand, by Figure 25(3), $x_2 \notin \{x_1, y_1, v, v_3, v_4, v_6\}$ and $y_2 \notin \{x_1, y_1, v, v_4, v_6, v_7\}$. Thus, $x_2 = v_7$ or $y_2 = v_3$. Indeed, by Figure 25(3) and the 1-planarity of H, at most one of $x_2 = v_7$ and $y_2 = v_3$ can happen. By symmetry in Figure 25(3), we assume that $y_2 = v_3$ (see Figure 26(1)). Note that both $\{x_1, v_4\}$ and $\{y_1, v_6\}$ are edges of H. To see this, first recall that $d_H(v_5) \leq 8$. So, v_5 has no other neighbors than those shown in Figure 26(1). Hence, v_4 and x_1 are consecutive neighbors of v_5 in H, and so are v_6 and y_1. In turn, by Condition 2 in Lemma 4.2, both $\{x_1, v_4\} \in E_H$ and $\{y_1, v_6\} \in E_H$.

Now, edge $\{v_4, x_1\}$ crosses no edge in H by Condition 2 in Lemma 4.2. Also, edge $\{v_5, x_2\}$ crosses no edge in H by Statement 2 in Corollary 4.5. Thus, as can be seen from Figure 26(1), H remains to be a 1-plane embedding even if we delete v_5 and v_4, merge v and x_1 along the two original edges $\{v, v_4\}$ and $\{v_4, x_1\}$, and merge v_6 and x_2 along the two original edges $\{v_6, v_5\}$ and $\{v_5, x_2\}$; hence, $(v_5, v_4, v, x_1, v_6, x_2)$ is a useful sextuple for v.

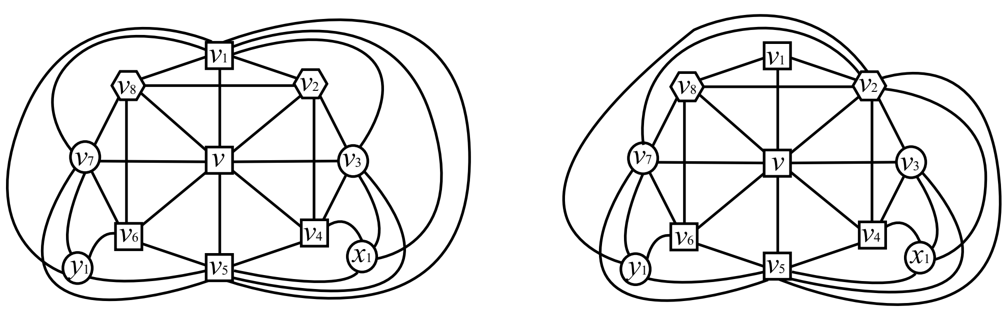
Case 2.5.2: $x_2 = y_2$ and $x_2 \notin N_H(v)$ (see Figure 26(2)). As in Case 2.5.1, we can show that both $\{x_1, v_4\} \in E_H$ and $\{y_1, v_6\} \in E_H$, and that $(v_5, v_4, v, x_1, v_6, x_2)$ is a useful sextuple for v.



(1) (2)

Figure 26: (1) Vertex v and its neighbors in K_v when $S_v^{\text{in}} \cup S_v^{\text{du}} = \{\{v_6, v_8\}, \{v_8, v_2\}, \{v_2, v_4\}\}$, $\{e_1, e_2\} \cap E_H = \emptyset$, $d_H(v_8) \geq 9$, $d_H(v_2) \geq 9$, $\{v_3, v_5\} \in E_H$, $\{v_7, v_5\} \in E_H$, and edges $\{v_3, v_5\}$ and $\{v_7, v_5\}$ cross edges $\{x_1, x_2\}$ and $\{y_1, y_2\}$ in H respectively such that $y_2 = v_3$. (2) Same as (1) except $x_2 = y_2 \notin N_H(v)$.

Case 2.5.3: $x_2 = y_2$ and $x_2 \in N_H(v)$. Then, as can be seen from Figure 25(3), $x_2 \in \{v_1, v_2, v_8\}$. If $x_2 = v_1$, then as can be seen from Figure 27(1), $d_H(v_1) = 8$ and edge $\{v_1, v_7\}$ crosses no edge in H by Statement 2 in Corollary 4.5, and in turn H remains to be a 1-plane embedding even if we delete v and v_1, merge v_2 and v_7 along the two original edges $\{v_2, v_1\}$ and $\{v_1, v_7\}$, and merge v_4 and x_8 along the two original edges $\{v_4, v\}$ and $\{v, v_8\}$; hence, $(v_1, v_2, v_7, v_4, v_8)$ is a favorite quintuple for v. Otherwise, $x_2 = v_2$ or $x_2 = v_8$. By symmetry in Figure 25(3), we assume that $x_2 = v_2$ (see Figure 27(2)). As can be seen from Figure 27(2), $d_H(v_5) = 8$ and edge $\{v_5, y_1\}$ crosses no edge in H by Statement 2 in Corollary 4.5, and in turn H remains to be a 1-plane embedding even if we delete v_5 and v, merge v_2 and v_6 along the two original edges $\{v_2, v\}$ and $\{v, v_6\}$, and merge v_4 and y_1 along the two original edges $\{v_4, v_5\}$ and $\{v_5, y_1\}$; hence, $(v_5, v, v_2, v_6, v_4, y_1)$ is a useful sextuple for v.



(1) (2)

Figure 27: (1) Vertex v and its neighbors in K_v when $S_v^{\text{in}} \cup S_v^{\text{du}} = \{\{v_6, v_8\}, \{v_8, v_2\}, \{v_2, v_4\}\}$, $\{e_1, e_2\} \cap E_H = \emptyset$, $d_H(v_8) \geq 9$, $d_H(v_2) \geq 9$, $\{v_3, v_5\} \in E_H$, $\{v_7, v_5\} \in E_H$, and edges $\{v_3, v_5\}$ and $\{v_7, v_5\}$ cross edges $\{x_1, x_2\}$ and $\{y_1, y_2\}$ in H respectively such that $x_2 = y_2 = v_1$. (2) Same as (1) except $x_2 = y_2 = v_2$.

5.4 The Case Where $d_H(v) = 8$ and $|S_v^{\text{in}}| + |S_v^{\text{du}}| = 4$

This case is complicated but not as much as the case in Section 5.3. Since $|S_v^{\text{in}}| + |S_v^{\text{du}}| = 4$, Fact 5.1 guarantees that there is a $v_i \in N_H(v)$ such that $S_v^{\text{in}} \cup S_v^{\text{du}}$ consists of edges $\{v_i, f_s^2(v_i)\}$, $\{f_s^2(v_i), f_s^4(v_i)\}$, $\{f_s^4(v_i), f_s^6(v_i)\}$, and $\{f_s^6(v_i), v_i\}$. By a relabeling if necessary, we can assume that $v_i = v_2$ (see Figure 28).

We distinguish three cases as follows.

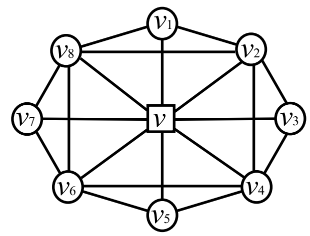


Figure 28: Vertex v and its neighbors in K_v when $S_v^{\text{in}} \cup S_v^{\text{du}} = \{\{v_2, v_4\}, \{v_4, v_6\}, \{v_6, v_8\}, \{v_8, v_2\}\}$.

Case 1: Both $d_H(v_4) \leq 8$ and $d_H(v_8) \leq 8$. Consider the following four possible edges: $e_1 = \{v_2, v_5\}$, $e_2 = \{v_3, v_6\}$, $e_3 = \{v_6, v_1\}$, and $e_4 = \{v_7, v_2\}$. Each of these edges may or may not be an edge of H. So, we illustrate them by broken edges in Figure 29.

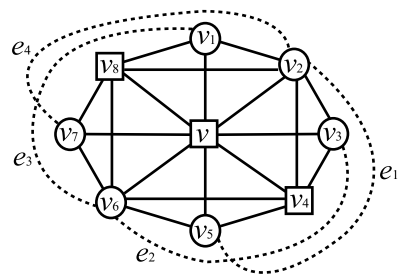


Figure 29: The four possible edges $e_1, \ldots, e_4$ when $S_v^{\text{in}} \cup S_v^{\text{du}} = \{\{v_2, v_4\}, \{v_4, v_6\}, \{v_6, v_8\}, \{v_8, v_2\}\}$.

We further distinguish three cases (one of them must occur) as follows.

Case 1.1: $\{e_1, e_3\} \cap E_H = \emptyset$ or $\{e_2, e_4\} \cap E_H = \emptyset$. By symmetry in Figure 29, we assume that $\{e_1, e_3\} \cap E_H = \emptyset$. Then, as can be seen from Figure 29, $(v_4, v_8, v_2, v_5, v_1, v_6)$ is a desired sextuple for v.

Case 1.2: $\{e_1, e_2\} \subseteq E_H$ or $\{e_3, e_4\} \subseteq E_H$. By symmetry in Figure 29, we assume that $\{e_1, e_2\} \subseteq E_H$. Then, since e_1 and e_2 have to cross each other in H (see Figure 29), Condition 3 in Lemma 4.2 guarantees that both $\{v_2, v_6\} \in E_H$ and $\{v_3, v_5\} \in E_H$ (see Figure 30(1)). We yet further distinguish two cases as follows.

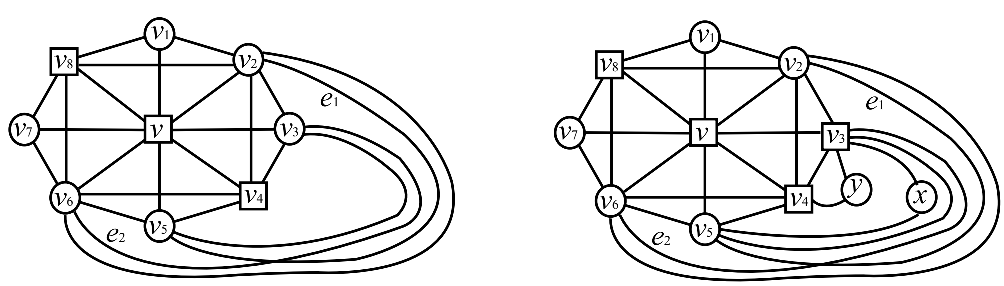


Figure 30: (1) Vertex v and its neighbors in K_v when $S_v^{\text{in}} \cup S_v^{\text{du}} = \{\{v_2, v_4\}, \{v_4, v_6\}, \{v_6, v_8\}, \{v_8, v_2\}\}$ and $\{e_1, e_2\} \subseteq E_H$. (2) Same as (1), but in addition $d_H(v_3) \leq 8$.

Case 1.2.1: $d_H(v_3) \leq 8$ or $d_H(v_5) \leq 8$. By symmetry in Figure 30(1), we assume that $d_H(v_3) \leq 8$. Then (see Figure 30(2)), since $d_H(v_3) \geq 7$ by the criticality of v, we can find the vertices $x \in N_H(v_3)$ and $y \in N_H(v_3)$ such that v_5 and x appear around v_3 consecutively in H in this order clockwise, and v_4 and y appear around v_3 consecutively in H in this order counterclockwise. Note that both $\{v_5, x\} \in E_H$ and $\{v_4, y\} \in E_H$ by Condition 2 in Lemma 4.2. If $d_H(v_3) = 7$, then

x and y are consecutive neighbors of v_3 in H and hence at least one of edges $\{v_3, x\}$ and $\{v_3, y\}$ crosses no edge in H by Statement 2 in Corollary 4.5. Moreover, if $d_H(v_3) = 7$ and edge $\{v_3, x\}$ crosses no edge in H, then as can be seen from Figure 30(2), (v_3, v_2, x) is a useful triple for v. Similarly, if $d_H(v_3) = 7$ and edge $\{v_3, y\}$ crosses no edge in H, then (v_3, v_2, y) is a useful triple for v. So, we are done when $d_H(v_3) = 7$. Thus, assume that $d_H(v_3) = 8$. Let z be the vertex in $N_H(v_3) - \{y, v_4, v, v_2, v_6, v_5, x\}$. By the 3-connectivity of H and our choice of x and y, vertices x, z, y appear around v_3 consecutively in H in this order clockwise. In turn, by Statement 2 in Corollary 4.5, at least one of edges $\{v_3, x\}$ and $\{v_3, z\}$ crosses no edge in H. If $\{v_3, x\}$ crosses no edge in H, then as can be seen from Figure 30(2), H remains to be a 1-plane embedding even if we delete v_3 and v_4, merge y and v along the two original edges $\{y, v_4\}$ and $\{v_4, v\}$, and merge x and v_2 along the two original edges $\{x, v_3\}$ and $\{v_3, v_2\}$. Thus, (v_3, v_4, v, y, v_2, x) is a useful sextuple for v. Similarly, if $\{v_3, z\}$ crosses no edge in H, then (v_3, v_4, v, y, v_2, z) is a useful sextuple for v.

Case 1.2.2: Both $d_H(v_3) \geq 9$ and $d_H(v_5) \geq 9$. Then, as can be seen from Figure 28, Condition 3 in the definition of reducibility of v guarantees that $d_H(v_2) \leq 8$ or $d_H(v_6) \leq 8$. If $d_H(v_2) \leq 8$, then as can be seen from Figure 30(1), $(v_2, v_1, v_3, v_4, v_8)$ is a favorite quintuple for v. Similarly, if $d_H(v_6) \leq 8$, then $(v_6, v_5, v_7, v_4, v_8)$ is a favorite quintuple for v.

Case 1.3: $\{e_1, \ldots, e_4\} \cap E_H = \{e_1, e_4\}$ or $\{e_1, \ldots, e_4\} \cap E_H = \{e_2, e_3\}$. By symmetry in Figure 29, we assume that $\{e_1, \ldots, e_4\} \cap E_H = \{e_1, e_4\}$. Then, as can be seen from Figure 31, $(v_4, v_8, v_3, v_6, v_1)$ is a desired quintuple for v.

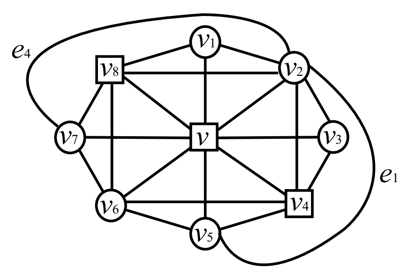


Figure 31: Vertex v and its neighbors in K_v when $S_v^{\text{in}} \cup S_v^{\text{du}} = \{\{v_2, v_4\}, \{v_4, v_6\}, \{v_6, v_8\}, \{v_8, v_2\}\}$ and $\{e_1, \ldots, e_4\} \cap E_H = \{e_1, e_4\}$.

Case 2: Both $d_H(v_2) \leq 8$ and $d_H(v_6) \leq 8$. Similar to Case 1 (by symmetry in Figure 28).

Case 3: At least one of v_4 and v_8 has degree ≥ 9 in H, and at least one of v_2 and v_6 has degree ≥ 9 in H. By symmetry in Figure 28, we assume that $d_H(v_8) \geq 9$ and $d_H(v_2) \geq 9$. Then, as can be seen from Figure 28, Condition 3 in the definition of reducibility of v guarantees that $d_H(v_1) \leq 8$, $d_H(v_4) \leq 8$, $d_H(v_5) \leq 8$, and $d_H(v_6) \leq 8$ (see Figure 32(1)). As in Case 2 in Section 5.3, consider the following two possible edges: $e_1 = \{v_2, v_6\}$ and $e_2 = \{v_4, v_8\}$. Either edge may or may not be an edge of H. So, we illustrate them by broken edges in Figure 30.

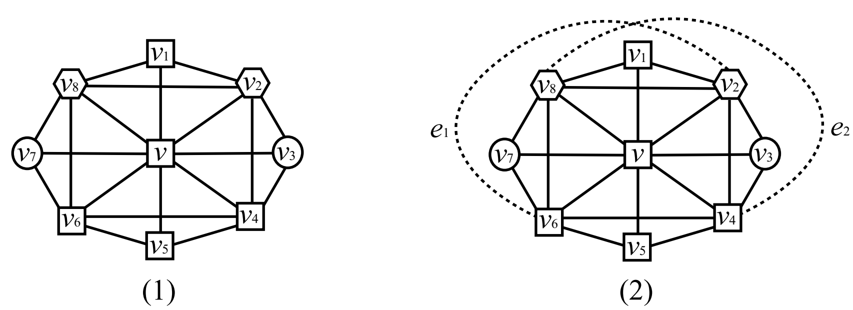


Figure 32: (1) Vertex v and its neighbors in K_v when $S_v^{\text{in}} \cup S_v^{\text{du}} = \{\{v_2, v_4\}, \{v_4, v_6\}, \{v_6, v_8\}, \{v_8, v_2\}\}$, $d_H(v_8) \geq 9$, and $d_H(v_2) \geq 9$. (2) The two possible edges e_1 and e_2 when $S_v^{\text{in}} \cup S_v^{\text{du}} = \{\{v_2, v_4\}, \{v_4, v_6\}, \{v_6, v_8\}, \{v_8, v_2\}\}$, $d_H(v_8) \geq 9$, and $d_H(v_2) \geq 9$.

If both $e_1 \in E_H$ and $e_2 \in E_H$, then as in Case 2.1 in Section 5.3, we can show that $\{v_1, v_3, v_7\}$ is a mergable triple for v.

If $\{e_1, e_2\} \cap E_H = \{e_1\}$, then as in Case 2.2.1 in Section 5.3, we can show that $(v_6, v_5, v_7, v_4, v_8)$ is a favorite quintuple for v. Similarly (by symmetry in Figure 32(2)), if $\{e_1, e_2\} \cap E_H = \{e_2\}$, $(v_4, v_3, v_5, v_2, v_6)$ is a favorite quintuple for v.

If $\{e_1, e_2\} \cap E_H = \emptyset$, then the arguments in Case 2.5 in Section 5.3 apply here (without any modification), because the arguments there remain valid even if $\{v_4, v_6\} \in E_H$.

6 NP-Completeness of 4-Colorability

Since planar graphs are special 1-planar graphs and it is NP-complete to decide whether a given planar graph is 3-colorable [12], it is also NP-complete to decide whether a given 1-planar graph is 3-colorable. We here show that it is also NP-complete to decide whether a given 1-planar graph is 4-colorable.

Theorem 6.1 *It is NP-complete to decide whether a given 1-planar graph is 4-colorable.*

Proof. By reduction from the problem of deciding whether a given planar graph is 3-colorable. Let $G = (V, E)$ be a planar graph. It takes linear time to embed G in the plane. Let G' be the graph obtained from G as follows. Add a new vertex v_{new} and put it in the outer face of G. Further draw an edge from v_{new} to each original vertex of G, letting edges cross if necessary (but each pair of edges may cross only once and there is no point at which three or more edges cross). This completes the construction of G'. Obviously, G is 3-colorable if and only if G' is 4-colorable. However, G' may not be 1-planar. To convert G' to a 1-planar graph, we use the following gadget H:

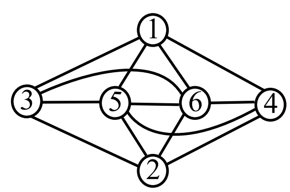


Figure 33: The gadget H.

We call vertices 1 through 4 of H the *corners* of H. Vertices 1 and 2 are *opposite* corners, and so are vertices 3 and 4. H is a 1-planar graph with the following properties:

- In any 4-coloring of H, each pair of opposite corners are forced to have the same color, and the two pairs of opposite corners have different colors.
- Any assignment of colors to vertices 1 through 4 such that each pair of opposite corners have the same color and the two pairs of corners have different colors, extends to a 4-coloring of all vertices of H.

We obtain a 1-planar graph G'' from G' as follows. For each point at which two edges e_1 and e_2 of G' cross, replace the point with a copy of H in such a way that a pair of opposite corners of the copy appear on e_1 and the other pair of opposite corners appear on e_2. Here, the copies used to replace the crossing points on each edge $\{u, w\}$ of G' must be located as shown in the following figure (that is, certain corners of the copies should be identified):

Obviously, the above replacement results in a 1-planar graph G''. It remains to show that G' is 4-colorable if and only if G'' is 4-colorable.

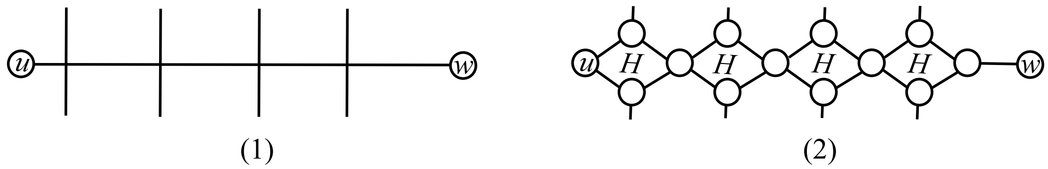


Figure 34: (1) An edge $\{u, w\}$ with four crossings. (2) After the replacement. Here, if $v_{new} \in \{u, w\}$, then $u = v_{new}$.

Suppose that G' has a 4-coloring. We obtain a 4-coloring of G'' as follows. Each vertex of G'' that is also a vertex of G' inherits its color from G'. The crucial point is that whenever two edges cross in G', one of them must be incident to v_{new} and hence both endpoints of the other edge have colors different from that of v_{new}. Because of this crucial point and the second property of H above, for each edge $\{u, w\}$ of G' with at least one crossing (see Figure 34(1)), we can propagate the color of u to the corners of the copies of H that appear on the edge $\{u, w\}$ of G' (see Figure 34(2)). In this way, the 4-coloring of G' extends to a 4-coloring of G''.

Conversely, suppose that G'' has a 4-coloring. Consider an edge $\{u, w\}$ of G'. If $\{u, w\}$ crosses no other edge of G', then $\{u, w\}$ is also an edge of G'' and hence u and w have different colors. Otherwise, by the first property of H above, u and w must have different colors. Thus, the 4-coloring of G'' restricted to those vertices of G' is a 4-coloring of G'. $\square$

It is natural to consider the problem of deciding whether a given 1-planar graph is 5-colorable. Unfortunately, we still do not know whether this problem is NP-complete. This is an open question.

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